

Quantum Field Theory - Lecture Notes

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1 Week 1

1.1 Lecture 1: Algebraic Methods in Quantum Mechanics

In this lecture we develop the algebraic method for solving the quantum harmonic oscillator (QHO), and we use it to motivate the general axiomatic structure of ladder-operator algebras that recurs throughout the course, most notably in the treatment of angular momentum later in this section.

The quantum harmonic oscillator

The Hamiltonian of the QHO is

$$H = \frac{1}{2} \frac{\hat{p}^2}{m} + \frac{1}{2} m \omega^2 \hat{x}^2, \quad [\hat{x}, \hat{p}] = i\hbar. \quad (1)$$

Our goal is to find the eigenvalues and eigenvectors of H using only the algebra of the *ladder operators*, without directly solving the Schrödinger differential equation. We will show that

$$H = \hbar\omega \left(a^\dagger a + \frac{1}{2} \right) = \hbar\omega \left(N + \frac{1}{2} \right), \quad (2)$$

where $N = a^\dagger a$ is the *number operator*. Rewriting the Hamiltonian in a suggestive form,

$$H = \hbar\omega \left(\frac{\hat{p}^2}{2\hbar m} + \frac{1}{2} \frac{m\omega}{\hbar} \hat{x}^2 \right). \quad (3)$$

We define the ladder operators

$$a = \sqrt{\frac{m\omega}{2\hbar}} \left(\hat{x} + \frac{i}{m\omega} \hat{p} \right), \quad a^\dagger = \sqrt{\frac{m\omega}{2\hbar}} \left(\hat{x} - \frac{i}{m\omega} \hat{p} \right). \quad (4)$$

By construction $a^\dagger$ is the Hermitian conjugate of a . Adding and subtracting these two relations and solving for $\hat{x}$ and $\hat{p}$ gives

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}} (a + a^\dagger), \quad \hat{p} = i\sqrt{\frac{m\omega\hbar}{2}} (a^\dagger - a). \quad (5)$$

Our Minimal Axioms

1. $[a, a^\dagger] = \mathbb{I}$.
2. Define the number operator $N = a^\dagger a$; it is Hermitian.
3. There exists a normalized state $|0\rangle$ such that $a|0\rangle = 0$, i.e. the Hilbert space has a state of minimal (bounded-below) energy.
4. For every $|\psi\rangle \in \mathcal{H}$, where $\mathcal{H}$ is a positive-definite Hilbert space, $\langle\psi|\psi\rangle \geq 0$, with equality if and only if $|\psi\rangle = 0$.

From the fundamental commutator one derives

$$[N, a] = -a, \quad [N, a^\dagger] = a^\dagger, \quad (6)$$

so that if $|n\rangle$ is an eigenstate of N with eigenvalue n , then $N(a|n\rangle) = (n-1)(a|n\rangle)$ and $N(a^\dagger|n\rangle) = (n+1)(a^\dagger|n\rangle)$: that is, $a|n\rangle$ and $a^\dagger|n\rangle$ are also eigenstates of N , with eigenvalues $n-1$ and $n+1$ respectively.

Let $|n\rangle$ be normalized eigenstates, $N|n\rangle = n|n\rangle$. Then

$$\langle\psi|\psi\rangle = \langle n|N|n\rangle = n \geq 0, \quad (7)$$

where $|\psi\rangle \equiv a|n\rangle$, so that $n \geq 0$ for every eigenvector. Writing $a|n\rangle = c_n|n-1\rangle$, normalization gives $\langle\psi|\psi\rangle = n = |c_n|^2$, so $c_n = \sqrt{n}$ (we discard $c_n = -\sqrt{n}$ since we have already shown all eigenvalues are non-negative, and we may fix the overall phase convention to be positive). Hence

$$a|n\rangle = \sqrt{n}|n-1\rangle. \quad (8)$$

Applying a repeatedly, $a^2|n\rangle = \sqrt{n(n-1)}|n-2\rangle$, and so on, giving

$$a^n|n\rangle = \sqrt{n!}|0\rangle. \quad (9)$$

The boundary condition of the theory — the existence of a lowest-energy state — is encoded entirely in the operators $a, a^\dagger$ and their algebra, namely the fact that they are Hermitian conjugates of one another.

The SU(2) Lie algebra

The generators J_x, J_y, J_z satisfy

$$[J_\mu, J_\nu] = i\varepsilon_{\mu\nu\sigma} J_\sigma. \quad (10)$$

We define ladder operators $J_\pm = J_x \pm iJ_y$, which satisfy the commutation relations

$$[J_z, J_\pm] = \pm J_\pm, \quad [J_+, J_-] = 2J_z. \quad (11)$$

We also define the Casimir operator

$$J^2 \equiv J_x^2 + J_y^2 + J_z^2 = J_z^2 + \frac{1}{2}(J_+ J_- + J_- J_+). \quad (12)$$

Since $[J^2, J_z] = 0$, the pair $\{J^2, J_z\}$ forms a Complete Set of Commuting Operators (CSCO) for this algebra.

This structure maps directly onto the minimal axioms above: the algebra itself realizes Axiom 1; since $J_z^\dagger = J_z$ and $J_\pm^\dagger = J_\mp$, the operator J^2 is Hermitian, realizing Axiom 2; and positivity of the norm, $\langle \psi | \psi \rangle \geq 0$, realizes Axiom 3.

Let $|j, m_j\rangle$ denote a state annihilated by raising, $J_+ |j, m_j\rangle = 0$, and with $J_z |j, m_j\rangle = m_j |j, m_j\rangle$. Because $[J^2, J_z] = 0$ we may diagonalize J^2 and J_z simultaneously:

$$J^2 |j, m\rangle = \lambda_j |j, m\rangle, \quad J_z |j, m\rangle = m |j, m\rangle. \quad (13)$$

Acting with $J_\pm$ shifts the J_z eigenvalue:

$$J_z (J_\pm |j, m\rangle) = (m \pm 1) (J_\pm |j, m\rangle). \quad (14)$$

1.2 Lecture 2: Representation Theory of SU(2)

Definition

A **Complete Set of Commuting Operators (CSCO)** is a set of mutually commuting operators whose common eigenvectors can be used as a basis to express any quantum state.

With $J_z |j, m\rangle = m |j, m\rangle$ and $J^2 |j, m\rangle = \lambda_j |j, m\rangle$, write $J_\pm |j, m\rangle = c_\pm |j, m \pm 1\rangle$. Let $|\psi_-\rangle = J_- |j, m\rangle = c_- |j, m-1\rangle$. Then

$$\langle \psi_- | \psi_- \rangle = \langle j, m | J_+ J_- |j, m\rangle = \langle j, m | (J^2 - J_z^2 + J_z) |j, m\rangle = \lambda_j - m^2 + m \geq 0. \quad (1)$$

Likewise, with $|\psi_+\rangle = J_+ |j, m\rangle$,

$$\langle \psi_+ | \psi_+ \rangle = \langle j, m | J_- J_+ |j, m\rangle = \lambda_j - m^2 - m \geq 0. \quad (2)$$

Plotting λ_j against m , conditions (1) and (2) require

$$\lambda_j \geq m^2 - m, \quad \lambda_j \geq m^2 + m, \quad (15)$$

which are both saturated at the extremal values of m :

$$\lambda_j = m_{\max}(m_{\max} + 1) = m_{\min}(m_{\min} - 1). \quad (16)$$

We impose one further condition: $m_{\max} - m_{\min} = n$ for some $n \in \mathbb{N}$ (raising by integer steps must connect the two extremal states). Substituting $m_{\max} = m_{\min} + n$,

$$(m_{\min} + n)(m_{\min} + n + 1) = m_{\min}(m_{\min} - 1) \implies 2m_{\min}(n + 1) = -n(n + 1) \implies m_{\min} = -\frac{n}{2}. \quad (17)$$

Hence $m_{\max} = n/2 \equiv j$, and

$$\lambda_j = j(j+1), \quad J_{\pm} |j, m\rangle = \sqrt{j(j+1) - m(m \pm 1)} |j, m \pm 1\rangle. \quad (18)$$

(In the position representation these matrix elements are related to the associated Legendre polynomials.)

The number of steps from $m_{\min}$ to $m_{\max}$ is $n+1 = 2j+1$, so the J_i are represented by $(2j+1) \times (2j+1)$ matrices, with matrix elements

$$\langle j, m' | J_z | j, m \rangle = m \delta_{m'm}, \quad \langle j, m' | J_{\pm} | j, m \rangle = \sqrt{j(j+1) - m(m \pm 1)} \delta_{m', m \pm 1}. \quad (19)$$

The group of relativity

SU(2) is the symmetry group of rotations, while relativity is governed by the symmetry group of boosts; rotations and boosts do not commute with one another. The group of relativity is $SO^+(3,1)$. Let

$$x^{\mu} \longrightarrow x'^{\mu} = \Lambda^{\mu}_{\nu} x^{\nu}, \quad \Lambda^{\mu}_{\nu} = \frac{\partial x'^{\mu}}{\partial x^{\nu}}. \quad (20)$$

We use the Minkowski metric with signature $\eta_{\mu\nu} = \text{diag}(+, -, -, -)$. The invariant interval is

$$\eta_{\mu\nu} x^{\mu} x^{\nu} = \eta_{\mu\nu} x'^{\mu} x'^{\nu}, \quad (21)$$

and substituting the transformation law,

$$\eta_{\mu\nu} x^{\mu} x^{\nu} = \eta_{\mu\nu} \Lambda^{\mu}_{\rho} \Lambda^{\nu}_{\sigma} x^{\rho} x^{\sigma} \implies \Lambda^{\mu}_{\rho} \eta_{\mu\nu} \Lambda^{\nu}_{\sigma} = \eta_{\rho\sigma}, \quad (22)$$

i.e. η is an invariant tensor, and $\Lambda^T \eta \Lambda = \eta$ is the defining condition of Lorentz orthogonality.

Axioms defining $SO^+(3,1)$

Axiom 1: Λ^{μ}_{ν} is a 4×4 matrix belonging to $O(3,1)$.

Axiom 2: $\det \Lambda = +1$ (this restricts $O(3,1)$ to $SO(3,1)$).

Axiom 3: $\Lambda^0_0 \geq 1$ (the “orthochronous” condition, restricting $SO(3,1)$ to $SO^+(3,1)$, needed to preserve time ordering).

Consider an infinitesimal transformation $\Lambda^{\mu}_{\nu} = \delta^{\mu}_{\nu} + \omega^{\mu}_{\nu}$, where δ^{μ}_{ν} is the Kronecker delta and ω^{μ}_{ν} is an infinitesimal deviation. Substituting into $\Lambda^T \eta \Lambda = \eta$ and keeping only first-order terms in ω ,

$$\eta_{\mu\nu} (\delta^{\mu}_{\rho} \delta^{\nu}_{\sigma} + \delta^{\mu}_{\rho} \omega^{\nu}_{\sigma} + \omega^{\mu}_{\rho} \delta^{\nu}_{\sigma}) = \eta_{\rho\sigma} \implies \omega_{\sigma\rho} = -\omega_{\rho\sigma}. \quad (23)$$

$$\omega_{\mu\nu} = -\omega_{\nu\mu} \quad (\omega_{\mu\nu} \text{ is antisymmetric in its indices}). \quad (24)$$

1.3 Lecture 3: The Lorentz Algebra

Let $U(\theta)$ be an element of the group, J be the algebra, and θ the parameter, so that $U(\theta) = \exp(-i\theta J)$. For the Lorentz group, any unitary representation on a Hilbert space has Hermitian generators $M_{\mu\nu} = -M_{\nu\mu}$ such that

$$U(\Lambda) = \exp\left(\frac{i}{2} \omega^{\mu\nu} M_{\mu\nu}\right). \quad (25)$$

Axiom 4

The Lie algebra of $SO(3,1)$ is determined by the commutators of $M_{\mu\nu}$.

The differential-operator (orbital) representation of the generators is

$$M_{\rho\sigma} = i(x_\rho \partial_\sigma - x_\sigma \partial_\rho), \quad (M_{\rho\sigma})^\mu{}_\nu = i(\delta^\mu{}_\rho \eta_{\sigma\nu} - \delta^\mu{}_\sigma \eta_{\rho\nu}). \quad (26)$$

Working out $[M_{\rho\sigma}, M_{\tau\lambda}]$ by direct computation on a test function (differentiating twice and antisymmetrizing) yields the Lorentz algebra

$$[M_{\rho\sigma}, M_{\tau\lambda}] = \eta_{\sigma\tau} M_{\rho\lambda} - \eta_{\rho\tau} M_{\sigma\lambda} + \eta_{\rho\lambda} M_{\sigma\tau} - \eta_{\sigma\lambda} M_{\rho\tau}. \quad (27)$$

It is convenient to define the rotation generators J_i and boost generators K_i by

$$J_i = \frac{1}{2} \varepsilon_{ijk} M_{jk}, \quad K_i = M_{0i}, \quad (28)$$

which satisfy

$$[J_i, J_j] = i\varepsilon_{ijk} J_k \quad (\text{rotation} + \text{rotation} = \text{rotation}), \quad (29)$$

$$[J_i, K_j] = i\varepsilon_{ijk} K_k \quad (\text{rotation} + \text{boost} = \text{boost}), \quad (30)$$

$$[K_i, K_j] = -i\varepsilon_{ijk} J_k \quad (\text{boost} + \text{boost} = \text{rotation}). \quad (31)$$

Free fields transform under Lorentz transformations according to representations of $SO(3, 1)$:

- **Scalar field** (spin 0): $\phi(x) \rightarrow \phi'(\Lambda x) = \phi(x)$, i.e. $M_{\mu\nu} = 0$ acting on the field.
- **Vector field** (spin 1): $A_\mu(x) \rightarrow \Lambda_\mu{}^\nu A_\nu(\Lambda^{-1}x)$, with generators acting as $(M_{\rho\sigma})_\mu{}^\nu = i(\delta^\nu{}_\rho \eta_{\sigma\mu} - \delta^\nu{}_\sigma \eta_{\rho\mu})$.
- **Spinor field** (fermions): $\psi(x) \rightarrow S(\Lambda)\psi(\Lambda^{-1}x)$, with $S(\Lambda) = \exp(-\frac{i}{4}\omega^{\mu\nu}\sigma_{\mu\nu})$. We will study this in detail once we reach the Dirac equation.

Decomposition into two $SU(2)$ algebras

Define the complex linear combinations of Lorentz generators

$$A_i = \frac{1}{2} (J_i + iK_i), \quad B_i = \frac{1}{2} (J_i - iK_i). \quad (32)$$

Note that A_i and B_i are not individually Hermitian; rather $A_i^\dagger = B_i$. They satisfy

$$[A_i, A_j] = i\varepsilon_{ijk} A_k, \quad [B_i, B_j] = i\varepsilon_{ijk} B_k, \quad [A_i, B_j] = 0. \quad (33)$$

The Lorentz algebra decomposes into two mutually commuting $SU(2)$ algebras:

$$SO(3, 1)_\mathbb{C} \cong SU(2)_L \otimes SU(2)_R. \quad (34)$$

(Proof left as an exercise.)

Writing the antisymmetric matrix of parameters as

$$\omega_{\mu\nu} = \begin{pmatrix} 0 & -\zeta_1 & -\zeta_2 & -\zeta_3 \\ \zeta_1 & 0 & \theta_3 & -\theta_2 \\ \zeta_2 & -\theta_3 & 0 & \theta_1 \\ \zeta_3 & \theta_2 & -\theta_1 & 0 \end{pmatrix}, \quad (35)$$

with θ_i the three *rotation* angles and ζ_i the three *boost* (rapidity) parameters,

[Correction/note:] the original notes labelled θ as the boost parameters and ζ as the rotation parameters; this is the reverse of the standard convention and inconsistent with the placement of these parameters in the $\omega_{\mu\nu}$ matrix itself, where the spatial-rotation block ($J_i = \frac{1}{2}\varepsilon_{ijk}M_{jk}$) sits in the lower-right 3×3 block and the boost block ($K_i = M_{0i}$) sits in the first row/column. The labelling has been corrected above.

the finite Lorentz transformation acting on the left- and right-handed $SU(2)$ factors is

$$\Lambda_L = \exp\left[\frac{i}{2}\theta \cdot \sigma - \frac{1}{2}\zeta \cdot \sigma\right], \quad \Lambda_R = \exp\left[\frac{i}{2}\theta \cdot \sigma + \frac{1}{2}\zeta \cdot \sigma\right]. \quad (36)$$

1.4 Lecture 4: Representations of the Lorentz Group

Since $SU(2)$ has three generators $\{J_1, J_2, J_3\}$, it is 3-dimensional as an algebra; its irreducible representations are labelled by j and have dimension $2j + 1$. A spin- j representation is irreducible: no representation of $SU(2)$ can be brought to block-diagonal form.

The orbital generators satisfy

$$M_{\mu\nu} = i(x_\mu\partial_\nu - x_\nu\partial_\mu), \quad [M_{\mu\nu}, M_{\rho\sigma}] = -i(\eta_{\mu\rho}M_{\nu\sigma} + \eta_{\nu\sigma}M_{\mu\rho} - \eta_{\mu\sigma}M_{\nu\rho} - \eta_{\nu\rho}M_{\mu\sigma}). \quad (37)$$

For the representation $(j_A, j_B) = (1/2, 0)$,

$$J_i = A_i + B_i = \frac{\sigma_i}{2}, \quad K_i = -i(A_i - B_i) = -\frac{i\sigma_i}{2}. \quad (38)$$

Clebsch–Gordan decompositions

Combining representations via tensor products (dimension counting on the left, spin content on the right):

$$2 \otimes 2 = 3 \oplus 1 \quad \left(\frac{1}{2} \otimes \frac{1}{2} = 1 \oplus 0\right), \quad (39)$$

$$4 \otimes 2 = 5 \oplus 3 \quad \left(\frac{3}{2} \otimes \frac{1}{2} = 2 \oplus 1\right), \quad (40)$$

$$5 \otimes 3 = 7 \oplus 5 \oplus 3 \quad (2 \otimes 1 = 3 \oplus 2 \oplus 1). \quad (41)$$

For $(j_1, j_2) = (3/2, 1/2) = 2 \oplus 1$: J_i are the generators of the $3/2 \otimes 1/2$ representation and $K_i = -iJ_i$, giving the angular-momentum (multipole) representation

$$\Lambda = \exp[i\theta \cdot \mathbf{J} + \zeta \cdot \mathbf{J}]. \quad (42)$$

Since the Lorentz algebra involves *two* copies of $SU(2)$, we must ask how to define a Casimir operator. Define

$$A^2 = \frac{1}{4}(J^2 - K^2) + \frac{i}{4}(\underbrace{\mathbf{J} \cdot \mathbf{K} + \mathbf{K} \cdot \mathbf{J}}_{\text{helicity}}), \quad B^2 = \frac{1}{4}(J^2 - K^2) - \frac{i}{4}(\mathbf{J} \cdot \mathbf{K} + \mathbf{K} \cdot \mathbf{J}). \quad (43)$$

Both A^2 and B^2 are Casimir invariants (constants), which means that their constituent pieces $J^2 - K^2$ and $\mathbf{J} \cdot \mathbf{K} + \mathbf{K} \cdot \mathbf{J}$ are *separately* conserved. A few remarks worth noting:

- The $\mathbf{J} \cdot \mathbf{K}$ (spin–orbit-like) coupling term is, in general, not separately conserved outside of this special combination.
- *Helicity* is the projection of the angular momentum along the direction of motion (momentum).
- The momentum operators $\hat{P}^\mu$ and $\hat{P}^\mu \hat{P}_\mu \sim m^2$ are the remaining Casimir-type invariants of the full Poincaré algebra (translations together with Lorentz transformations).

Finally, for a scalar test function f , one can verify the Lorentz algebra directly by expanding

$$[M_{\mu\nu}, M_{\rho\sigma}]f = M_{\mu\nu}(M_{\rho\sigma}f) - M_{\rho\sigma}(M_{\mu\nu}f) \quad (44)$$

using $M_{\rho\sigma} = i(x_\rho\partial_\sigma - x_\sigma\partial_\rho)$ and the identity $\partial_\nu x_\rho = \eta_{\nu\rho}$. Carrying out the (somewhat lengthy) product-rule expansion and its mirror image for $M_{\rho\sigma}M_{\mu\nu}f$, and subtracting, reproduces exactly the general commutation relation already quoted above for $[M_{\rho\sigma}, M_{\tau\lambda}]$ (with indices relabelled), confirming the consistency of the algebra.

2 Week 2

2.1 Lecture 5: Units and Constants

$$c = 3 \times 10^8 \text{ m/s} = [LT^{-1}], \quad \hbar = \frac{6.6 \times 10^{-34} \text{ J} \cdot \text{s}}{2\pi} = [ML^2T^{-1}], \quad G_N = [M^{-1}L^3T^{-2}]. \quad (45)$$

Natural units

We work in natural units, $c = 1$, $\hbar = 1$. Since $E = mc^2$ and $c = 1$ is dimensionless, mass carries the same units as energy. Since $E = \hbar\omega$ and $\hbar = 1$, angular frequency ω also carries the same units as energy. Setting $c = 1$ forces $[LT^{-1}] = 1$, i.e. $[L] = [T]$ — a consequence that underlies the four-vector formulation of relativity, in which space and time share a common unit. Hence

$$[\text{Energy}] = [T]^{-1} = [L]^{-1}. \quad (46)$$

The four-momentum is $p^\mu = (E/c, p_x, p_y, p_z)$, but once $c = 1$ this becomes $[E] = [p_x]$: in natural units there is only *one* independent dimension, energy.

A linear-independence check

As an aside, consider whether $(0, 1, -1)$, $(1, 2, -1)$, $(-1, 3, -2)$ are linearly independent and therefore span $\mathbb{R}^3$. Computing the determinant of the matrix formed by these three vectors as rows,

$$\det \begin{pmatrix} 0 & 1 & -1 \\ 1 & 2 & -1 \\ -1 & 3 & -2 \end{pmatrix} = 0(-4 + 3) - 1(-2 - 1) - 1(3 + 2) = 0 + 3 - 5 = -2 \neq 0, \quad (47)$$

so the three vectors are linearly independent and therefore span $\mathbb{R}^3$.

Planck units and Okun's cube

$$\text{Planck mass: } M_P = \sqrt{\frac{\hbar c}{G_N}} = 2.18 \times 10^{-8} \text{ kg}, \quad (48)$$

$$\text{Planck length: } \ell_P = \sqrt{\frac{\hbar G_N}{c^3}} = 1.6 \times 10^{-35} \text{ m}, \quad (49)$$

$$\text{Planck time: } t_P = \sqrt{\frac{\hbar G_N}{c^5}} = 3 \times 10^{-44} \text{ s}. \quad (50)$$

Okun's cube organizes physical theories along the three axes $(G_N, 1/c, \hbar)$:

- $(0, 0, 0)$: classical (Newtonian, non-relativistic) limit.
- $(G_N, 0, 0)$: Newtonian gravity.

- $(0, 1/c, 0)$: special relativity.
- $(0, 0, \hbar)$: quantum mechanics.
- $(0, 1/c, \hbar)$: quantum field theory / relativistic quantum mechanics.
- $(G_N, 1/c, 0)$: general relativity.
- $(G_N, 0, \hbar)$: non-relativistic quantum gravity.
- $(G_N, 1/c, \hbar)$: quantum gravity (not yet achieved).

The Schwarzschild radius is $R_s = 2G_N M/c^2$. With $c = 1$ and $[G_N] = [M^{-2}]$,

$$R_s = 2[M^{-2}][M] \implies [R_s] = [M^{-1}], \quad (51)$$

which is self-consistent, since $[L] = [M^{-1}]$ in natural units.

[Correction/note:] One must *not* additionally set $G_N = 1$ on top of $c = \hbar = 1$: doing so would force $[R_s] \sim [M]$, which contradicts the dimensional analysis above. Natural units fix only $c = \hbar = 1$, leaving G_N (equivalently, the Planck mass) as a genuine dimensionful scale.

Buckingham's Π theorem

If a physical system is described in terms of N physical quantities built from M independent basic dimensions, then the system can equivalently be described using $p = N - M$ independent dimensionless quantities.

2.2 Lectures 6–7: Hamiltonian Mechanics and the Klein–Gordon Equation

Let there exist canonical coordinates q_i with conjugate momenta p_i , related by the fundamental Poisson bracket $\{q_i, p_j\} = \delta_{ij}$. The Hamiltonian equations of motion are

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}. \quad (52)$$

For phase-space functions f, g , the *Poisson bracket* is

$$\{f, g\} = \sum_i \left[\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right]. \quad (53)$$

Poisson brackets are bilinear, antisymmetric, and satisfy the Jacobi identity; consequently, Poisson brackets form a Lie bracket.

Define the phase-space vector $\xi = (q, p)^\top$. Then $\dot{\xi} = \{\xi, H\}$. Indeed, if $\xi_i = q_i$, then, using $\partial q_i / \partial p_j = 0$ and $\partial q_i / \partial q_j = \delta_{ij}$,

$$\dot{\xi}_i = \{q_i, H\} = \sum_j \left[\frac{\partial q_i}{\partial q_j} \frac{\partial H}{\partial p_j} - \frac{\partial H}{\partial q_j} \frac{\partial q_i}{\partial p_j} \right] = \frac{\partial H}{\partial p_i} = \dot{q}_i, \quad (54)$$

consistent with Hamilton's equations; similarly, for $\xi_i = p_i$, $\dot{\xi}_i = \{p_i, H\} = -\partial H / \partial q_i = \dot{p}_i$. If $\mathbf{q} \in \mathbb{R}^n$, then $\xi \in \Pi = \mathbb{R}^n \times \mathbb{R}^n = \mathbb{R}^{2n}$, the *phase space*.

From the Schrödinger equation to Klein–Gordon

The quantum-mechanical equation of motion is the Schrödinger equation,

$$i\hbar \partial_t |\psi\rangle = H |\psi\rangle, \quad H = \frac{\hat{p}^2}{2m} + V(x) = -\frac{1}{2m} \partial_x^2 + V(x). \quad (55)$$

For a relativistic theory the dispersion relation is $E^2 = p^2 + m^2$. Using the operator replacements $\hat{p} = -i\partial_x$ and $\hat{E} = i\partial_t$,

$$-\partial_t^2 \psi = (-\partial_x^2 + m^2) \psi. \quad (56)$$

The solution to this wave equation has the plane-wave form $\psi \sim e^{i(\omega t - kx)}$ with $\omega^2 = k^2 + m^2$. Defining light-cone coordinates $z_{\pm} = x \pm vt$ and writing the general solution as $\psi(x) = \psi(z_+) + \psi(z_-)$, one arrives at

$$\partial_t^2 \psi - \partial_x^2 \psi + m^2 \psi = 0 \quad (\text{the Klein–Gordon equation}). \quad (57)$$

In four-vector notation, with $x^\mu = (x^0, x^1, x^2, x^3) = (t, x, y, z)$, $\eta = \text{diag}(1, -1, -1, -1)$, and $x_\mu = (t, -x, -y, -z)$, the d'Alembertian is $\partial^\mu \partial_\mu = \partial_t^2 - \partial_x^2$, so that the Klein–Gordon equation is written covariantly as

$$\partial^\mu \partial_\mu \psi + m^2 \psi = 0. \quad (58)$$

Probability current and its conservation

Starting from the Schrödinger equation and its complex conjugate,

$$i\partial_t \psi = -\frac{1}{2m} \partial_x^2 \psi + V(x)\psi, \quad (1)$$

$$-i\partial_t \psi^* = -\frac{1}{2m} \partial_x^2 \psi^* + V(x)\psi^*, \quad (2)$$

forming $\psi^* \times (1) - \psi \times (2)$ and using $V(x)$ real (so the potential terms cancel), one finds

$$i\partial_t(\psi\psi^*) = -\frac{1}{2m} (\psi^* \partial_x^2 \psi - \psi \partial_x^2 \psi^*). \quad (59)$$

Defining the probability density $\rho = \psi\psi^* = |\psi|^2$ and the probability current

$$j = -\frac{1}{2mi} (\psi^* \partial_x \psi - \psi \partial_x \psi^*), \quad (60)$$

this becomes the **continuity equation**

$$\frac{\partial \rho}{\partial t} + \nabla \cdot j = 0. \quad (61)$$

Claim: $\partial^\mu j_\mu = 0$. By the Klein–Gordon equation, $(\partial^\mu \partial_\mu + m^2)\psi(x) = 0$ and likewise $(\partial^\mu \partial_\mu + m^2)\psi^*(x) = 0$. Forming $\psi^*(\partial^\mu \partial_\mu + m^2)\psi - \psi(\partial^\mu \partial_\mu + m^2)\psi^* = 0$, the mass terms cancel, leaving

$$\psi^* \partial^\mu \partial_\mu \psi - \psi \partial^\mu \partial_\mu \psi^* = 0 \implies \partial^\mu (\psi^* \partial_\mu \psi - \psi \partial_\mu \psi^*) = 0. \quad (62)$$

Defining the four-current

$$j^\mu = i(\psi^* \partial^\mu \psi - \psi \partial^\mu \psi^*), \quad (63)$$

we obtain

$$\partial_\mu j^\mu = 0 \quad (\text{the relativistic four-current is conserved}). \quad (64)$$

2.3 Lecture 8: Classical Field Theory and the Euler–Lagrange Equation

For a 1+1-dimensional system, the Lagrangian is $L = \int dx \mathcal{L}(\varphi, \partial_\mu \varphi, x^\mu)$; for a 3+1-dimensional system the action is

$$S = \int dt L = \int d^4x \mathcal{L}(\varphi, \partial_\mu \varphi, x^\mu). \quad (65)$$

Consider a field variation $\varphi(t, \mathbf{x}) \rightarrow \varphi'(t, \mathbf{x}) = \varphi(t, \mathbf{x}) + \delta\varphi(t, \mathbf{x})$. Then

$$\delta S = \int \left[\frac{\partial \mathcal{L}}{\partial \varphi} \delta\varphi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \delta(\partial_\mu \varphi) + \frac{\partial \mathcal{L}}{\partial x^\mu} \underbrace{\delta(x^\mu)}_{=0} \right] d^4x. \quad (66)$$

Using the fact that variation and differentiation commute, $\delta(\partial_\mu \varphi) = \partial_\mu(\delta\varphi)$, this becomes

$$\delta S = \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \varphi} \delta\varphi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \partial_\mu(\delta\varphi) \right]. \quad (67)$$

Integrating the second term by parts, $\int f'g = -\int fg' + [fg]$, gives

$$\delta S = \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \varphi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \right) \right] \delta\varphi + (\text{boundary term}). \quad (68)$$

By the variational principle, $\delta S = 0$ for arbitrary $\delta\varphi$; the boundary term is set to zero (though *not* in all cases — for topological field theories, boundary terms are very important). This yields the **Euler–Lagrange equation**, a *local* property of the field, not a global one:

$$\frac{\partial \mathcal{L}}{\partial \varphi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \right) = 0. \quad (69)$$

Example: φ^4 theory

Take

$$\mathcal{L} = \frac{1}{2}(\partial^\mu \varphi)(\partial_\mu \varphi) - \frac{1}{2}m^2\varphi^2 - \frac{\lambda}{4!}\varphi^4. \quad (70)$$

Then $\partial \mathcal{L} / \partial \varphi = -m^2\varphi - (\lambda/3!)\varphi^3$ and $\partial \mathcal{L} / \partial(\partial_\mu \varphi) = \partial^\mu \varphi$, so the Euler–Lagrange equation gives the equation of motion

$$\partial_\mu \partial^\mu \varphi + m^2\varphi + \frac{\lambda}{3!}\varphi^3 = 0 \quad (\text{Klein–Gordon equation with a quartic perturbation}). \quad (71)$$

Two coupled scalar fields and an internal $O(2)$ symmetry

Consider two real scalar fields with a cubic coupling,

$$\mathcal{L} = \frac{1}{2}(\partial^\mu \varphi_1)(\partial_\mu \varphi_1) - \frac{m_1^2}{2}\varphi_1^2 + \frac{1}{2}(\partial^\mu \varphi_2)(\partial_\mu \varphi_2) - \frac{m_2^2}{2}\varphi_2^2 - g\varphi_1\varphi_2^2. \quad (72)$$

The Euler–Lagrange equations of motion are

$$\partial^\mu \partial_\mu \varphi_1 + m_1^2\varphi_1 + g\varphi_2^2 = 0, \quad (73)$$

$$\partial^\mu \partial_\mu \varphi_2 + m_2^2\varphi_2 + 2g\varphi_1\varphi_2 = 0. \quad (74)$$

Consider the free ($g = 0$) theory with $m_1 = m_2 \equiv m$, and rotate to a new field basis $\varphi_i \rightarrow \chi_i$ via

$$\begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \equiv R(\theta) \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}. \quad (75)$$

In matrix form the free Lagrangian is

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi_1, \partial_\mu \varphi_2) \begin{pmatrix} \partial^\mu \varphi_1 \\ \partial^\mu \varphi_2 \end{pmatrix} - \frac{1}{2}(\varphi_1, \varphi_2) \begin{pmatrix} m_1^2 & 0 \\ 0 & m_2^2 \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}. \quad (76)$$

Under the rotation, the kinetic term picks up $R^\top R = \mathbb{1}$ (invariant for any θ), while the mass term picks up $R^\top \text{diag}(m_1^2, m_2^2) R$, which equals $\text{diag}(m_1^2, m_2^2)$ *only if* $m_1 = m_2$. Hence the Lagrangian is invariant under this internal rotation i.e. it possesses an $O(2)$ internal symmetry - if and only if the two masses are equal.

For an infinitesimal rotation, $\theta \ll 1$,

$$\begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} \approx \begin{pmatrix} 1 & -\theta \\ \theta & 1 \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \implies \chi_1 = \varphi_1 - \theta \varphi_2, \quad \chi_2 = \varphi_2 + \theta \varphi_1. \quad (77)$$

Writing $\chi_i = \varphi_i + \delta\varphi_i$, the infinitesimal transformation is

$$\delta\varphi_1 = -\theta\varphi_2, \quad \delta\varphi_2 = \theta\varphi_1. \quad (78)$$

This is the infinitesimal form of the $O(2)$ internal symmetry of the degenerate-mass theory ($m_1 = m_2$); by Noether's theorem, to be developed in a later lecture, this continuous symmetry will give rise to a conserved current.

3 Weeks 3 and 4

3.1 Lecture 9: Hamiltonian Formulation of Field Theory

$$\mathcal{L} = \mathcal{L}(q_i, \dot{q}_i, t), \quad p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i}.$$

Define the Hamiltonian $H(p_i, q_i, t)$, which is the Legendre transform of $\mathcal{L}$,

$$H(p_i, q_i, t) = \sum_i p_i \dot{q}_i - \mathcal{L}(q_i, \dot{q}_i, t). \quad (79)$$

For fields we have Lagrangian densities, $\mathcal{L} = \mathcal{L}(\phi_a, \partial_\mu \phi_a, x^\mu)$. Define the conjugate momenta (the field momentum density)

$$\pi_a = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi_a)} \longrightarrow \text{we can't write Poisson brackets, as there is a momentum density.} \quad (80)$$

$$H = \sum_a \pi_a \dot{\phi}_a - \mathcal{L}(\phi_a, \partial_\mu \phi_a, x^\mu) = H(\phi_a, \pi_a, \partial_i \phi_a, x^\mu). \quad (81)$$

Unlike Mechanics, where we replace $\dot{q}_i$ with p_i in the Hamiltonian, we don't replace $\partial_\mu \phi_a$ for fields. ONLY $\partial_0 \phi_a$.

$$\partial_\mu \phi_a = (\partial_0 \phi_a, \partial_i \phi_a), \quad i = 1, 2, 3.$$

So only $\partial_0 \phi_a$ is replaced by π_a in the Hamiltonian.

$$\frac{\partial H}{\partial \pi_a} = \dot{\phi}_a, \quad \frac{\partial H}{\partial \phi_a} = -\dot{\pi}_a \implies \text{Hamilton's equations of motion.} \quad (82)$$

$$dH = \frac{\partial H}{\partial \phi_a} d\phi_a + \frac{\partial H}{\partial \pi_a} d\pi_a + \frac{\partial H}{\partial(\partial_i \phi_a)} d(\partial_i \phi_a) + \frac{\partial H}{\partial x^\mu} dx^\mu, \quad (83)$$

$$\begin{aligned} dH &= d(\pi_a \dot{\phi}_a) - d\mathcal{L} \\ &= d\pi_a \dot{\phi}_a + \pi_a d(\dot{\phi}_a) - \frac{\partial \mathcal{L}}{\partial \phi_a} d\phi_a - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} d(\partial_\mu \phi_a) - \frac{\partial \mathcal{L}}{\partial x^\mu} dx^\mu. \end{aligned} \quad (84)$$

Comparing terms (splitting $\partial_\mu \phi_a$ into $\dot{\phi}_a + \partial_i \phi_a$),

$$= \left(\pi_a - \frac{\partial \mathcal{L}}{\partial \dot{\phi}_a} \right) d\dot{\phi}_a - \frac{\partial \mathcal{L}}{\partial \phi_a} d\phi_a - \frac{\partial \mathcal{L}}{\partial(\partial_i \phi_a)} d(\partial_i \phi_a) + \dot{\phi}_a d\pi_a - \frac{\partial \mathcal{L}}{\partial x^\mu} dx^\mu, \quad (85)$$

so that, comparing coefficients term by term,

$$\frac{\partial \mathcal{L}}{\partial \phi_a} = -\frac{\partial H}{\partial \phi_a}, \quad \frac{\partial \mathcal{L}}{\partial x^\mu} = -\frac{\partial H}{\partial x^\mu}, \quad \frac{\partial \mathcal{L}}{\partial(\partial_i \phi_a)} = -\frac{\partial H}{\partial(\partial_i \phi_a)}. \quad (86)$$

By the Euler–Lagrange equations of motion (ELOM),

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \phi_a} &= \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \right) = \partial_0 \left(\frac{\partial \mathcal{L}}{\partial(\partial_0 \phi_a)} \right) + \partial_i \left(\frac{\partial \mathcal{L}}{\partial(\partial_i \phi_a)} \right) \\ &= \dot{\pi}_a + \partial_i \left(\frac{\partial \mathcal{L}}{\partial(\partial_i \phi_a)} \right). \end{aligned} \quad (87)$$

$$-\frac{\partial H}{\partial \phi_a} = \dot{\pi}_a - \partial_i \left(\frac{\partial H}{\partial(\partial_i \phi_a)} \right) \implies \dot{\pi}_a = -\frac{\partial H}{\partial \phi_a} + \partial_i \left(\frac{\partial H}{\partial(\partial_i \phi_a)} \right). \quad (88)$$

Define the *functional derivative*

$$\frac{\delta F}{\delta \phi_a} \equiv \frac{\partial F}{\partial \phi_a} - \partial_i \left(\frac{\partial F}{\partial(\partial_i \phi_a)} \right). \quad (89)$$

Let F be any function on the phase space. Then

$$\dot{\phi}_a = \frac{\delta H}{\delta \pi_a}, \quad \dot{\pi}_a = -\frac{\delta H}{\delta \phi_a}. \quad (90)$$

Let $F = F(\phi_a, \partial_i \phi_a, \pi_a, \partial_i \pi_a, x^\mu)$ and define $\mathcal{F} = \int F d^3x$. Then

$$dF = \frac{\partial F}{\partial \phi_a} d\phi_a + \frac{\partial F}{\partial(\partial_i \phi_a)} d(\partial_i \phi_a) + \frac{\partial F}{\partial \pi_a} d\pi_a + \frac{\partial F}{\partial(\partial_i \pi_a)} d(\partial_i \pi_a) + \frac{\partial F}{\partial x^\mu} dx^\mu. \quad (91)$$

After integrating by parts and discarding the boundary term,

$$\begin{aligned} &\left(\frac{\partial F}{\partial \phi_a} - \partial_i \left(\frac{\partial F}{\partial(\partial_i \phi_a)} \right) \right) d\phi_a + \left(\frac{\partial F}{\partial \pi_a} - \partial_i \left(\frac{\partial F}{\partial(\partial_i \pi_a)} \right) \right) d\pi_a + \frac{\partial F}{\partial x^\mu} dx^\mu \\ &= \frac{\delta F}{\delta \phi_a} d\phi_a + \frac{\delta F}{\delta \pi_a} d\pi_a + \frac{\partial F}{\partial x^\mu} dx^\mu. \end{aligned} \quad (92)$$

$$\begin{aligned} \frac{d\mathcal{F}}{dt} &= \int d^3x \left[\frac{\delta F}{\delta \phi_a} \frac{d\phi_a}{dt} + \frac{\delta F}{\delta \pi_a} \frac{d\pi_a}{dt} \right] + \int d^3x \frac{\partial F}{\partial x^\mu} \frac{dx^\mu}{dt} \\ &= \int d^3x \left[\frac{\delta F}{\delta \phi_a} \frac{\delta H}{\delta \pi_a} - \frac{\delta F}{\delta \pi_a} \frac{\delta H}{\delta \phi_a} \right] + \partial_0 \mathcal{F}, \end{aligned} \quad (93)$$

$$\frac{d\mathcal{F}}{dt} = \int d^3x \{F, H\} + \partial_0 \mathcal{F}. \quad (94)$$

Let

$$\tilde{\phi}_a(\mathbf{k}, t) = \int d^3x e^{-i\mathbf{k}\cdot\mathbf{x}} \phi_a(\mathbf{x}, t), \quad \tilde{\pi}_a(\mathbf{k}, t) = \int d^3x e^{-i\mathbf{k}\cdot\mathbf{x}} \pi_a(\mathbf{x}, t) \quad (95)$$

[Correction/note:] these Fourier transforms are taken over the spatial slice only, using d^3x and $e^{-i\mathbf{k}\cdot\mathbf{x}}$; as the original notes remark, this is *not* a Lorentz-invariant measure.

be the spatial Fourier transforms of the field and its momentum density. Then

$$\begin{aligned} \{\tilde{\phi}_a(\mathbf{k}, t), \tilde{\pi}_b(\mathbf{k}', t)\} &= \int d^3x \left(\frac{\delta\tilde{\phi}_a}{\delta\phi_c} \frac{\delta\tilde{\pi}_b}{\delta\pi_c} - \frac{\delta\tilde{\phi}_a}{\delta\pi_c} \frac{\delta\tilde{\pi}_b}{\delta\phi_c} \right) \\ &= \int d^3x \delta_{bc} \delta_{ac} \left(e^{-i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{x}} \right) \\ &= \delta_{ab} \delta^{(3)}(\mathbf{k} + \mathbf{k}') \cdot (2\pi)^3, \end{aligned} \quad (96)$$

$$\{\tilde{\phi}_a, \tilde{\phi}_b\} = 0 = \{\tilde{\pi}_a, \tilde{\pi}_b\}. \quad (97)$$

3.2 Lecture 10: Noether's Theorem

Noether's Theorem

For every continuous global symmetry of the action, there exists a conserved current $\partial^\mu J_\mu = 0$.

Global Symmetry

For an infinitesimal field transformation $\phi_a \rightarrow \phi'_a = \phi_a + \delta\phi_a$, the transformation is said to be *global* if

$$\delta S(\phi_a, \delta\phi_a) = S(\phi_a + \delta\phi_a) - S(\phi_a) = \int d^4x \partial_\mu K^\mu, \quad \forall \phi_a, \quad (98)$$

for some K^μ that depends on ϕ_a and $\delta\phi_a$.

If the action is invariant under a transformation of the fields up to a boundary term, it is a global symmetry.

Let $\delta\phi_a$ be a continuous global symmetry of the action. Then

$$\delta S(\phi_a, \delta\phi_a) = \int d^4x \partial_\mu K^\mu, \quad (99)$$

$$\delta S[\phi_a, \delta\phi_a] = \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi_a} \delta\phi_a + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_a)} \delta(\partial_\mu \phi_a) \right]. \quad (100)$$

Since $\phi_a \rightarrow \phi_a + \delta\phi_a$, $\delta(\partial_\mu \phi_a) = \partial_\mu(\delta\phi_a)$, so

$$= \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi_a} \delta\phi_a + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_a)} \partial_\mu(\delta\phi_a) \right]. \quad (101)$$

Using

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_a)} \delta\phi_a \right) = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_a)} \right) \delta\phi_a + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_a)} \partial_\mu(\delta\phi_a), \quad (102)$$

and the Euler–Lagrange equations of motion to kill the first term,

$$\delta S(\phi_a, \delta\phi_a) = \int d^4x \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_a)} \delta\phi_a \right). \quad (103)$$

Hence, letting $\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \delta \phi_a \right) = \partial_\mu K^\mu$,

$$\underbrace{\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \delta \phi_a - K^\mu \right)}_{j^\mu} = 0. \quad (104)$$

Hence we have the conservation equation $\partial_\mu j^\mu = 0$.

$$\frac{\partial \rho}{\partial t} + \nabla \cdot j = 0 \quad \implies \quad \frac{d}{dt} \int d^3x \rho + \int d^3x \nabla \cdot j = 0. \quad (105)$$

Using the divergence theorem, the surface integral at infinity vanishes,

$$\implies \frac{dQ}{dt} = 0 \quad \implies \quad \text{charge is conserved for a conserved current } j^\mu. \quad (106)$$

Example: the complex scalar field

Consider

$$\mathcal{L} = (\partial^\mu \phi)^\dagger (\partial_\mu \phi) - m^2 \phi^\dagger \phi, \quad (107)$$

where $\phi = \frac{\phi_1 + i\phi_2}{\sqrt{2}} \Rightarrow \phi^\dagger = \frac{\phi_1 - i\phi_2}{\sqrt{2}}$, with ϕ_1, ϕ_2 real scalar fields. Let $\phi \rightarrow \phi' = e^{i\alpha} \phi$; infinitesimally $\phi' = (1 + i\alpha)\phi$, and $\phi^\dagger \rightarrow (\phi^\dagger)' = e^{-i\alpha} \phi^\dagger$, or $(\phi^\dagger)' = \phi^\dagger - i\alpha \phi^\dagger$. Then

$$\delta_S \phi = i\alpha \phi, \quad \delta_S \phi^\dagger = -i\alpha \phi^\dagger. \quad (108)$$

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta_S \phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^\dagger)} \delta_S \phi^\dagger = i\alpha \left[(\partial^\mu \phi)^\dagger \phi - \phi^\dagger (\partial^\mu \phi) \right]. \quad (109)$$

In terms of ϕ_1, ϕ_2 ,

$$j^\mu = \phi_1 \partial^\mu \phi_2 - \phi_2 \partial^\mu \phi_1, \quad Q = \int d^3x j^0 \longrightarrow Q \text{ is non-trivial only when } \phi \text{ is complex.} \quad (110)$$

3.3 Class 11: Space-Time Symmetries and the Energy-Momentum Tensor

Consider a Lagrangian density $\mathcal{L}(\phi_a, \partial_\nu \phi_a)$. Let $x^\mu \rightarrow x'^\mu = x^\mu + \varepsilon^\mu$, where ε^μ is a constant infinitesimal vector. Then

$$\phi_a(x) \rightarrow \phi'_a(x) = \phi_a(x - \varepsilon) = \phi_a(x) - \varepsilon^\mu \partial_\mu \phi_a(x) \quad \implies \quad \delta \phi_a = -\varepsilon^\mu \partial_\mu \phi_a(x). \quad (111)$$

$$\delta_S(\partial_\nu \phi_a) = \partial_\nu \phi_a(x - \varepsilon) - \partial_\nu \phi_a(x) = -\varepsilon^\mu \partial_\mu \partial_\nu \phi_a = \partial_\nu(\delta_S \phi_a). \quad (112)$$

$$\begin{aligned} \delta \mathcal{L} &= \frac{\partial \mathcal{L}}{\partial \phi_a} \delta \phi_a + \frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi_a)} \delta_S(\partial_\nu \phi_a) \\ &= -\varepsilon^\mu \left[\frac{\partial \mathcal{L}}{\partial \phi_a} \partial_\mu \phi_a + \frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi_a)} \partial_\mu \partial_\nu \phi_a \right]. \end{aligned} \quad (113)$$

$$\delta_S \mathcal{L} = -\varepsilon^\mu \partial_\mu \mathcal{L} = \partial_\mu (-\varepsilon^\mu \mathcal{L}) \equiv \partial_\mu K^\mu, \quad K^\mu = -\varepsilon^\mu \mathcal{L}. \quad (114)$$

The action is invariant under an infinitesimal translation of spacetime coordinates up to a boundary term: $\delta_S S = \int d^4x \partial_\mu K^\mu$.

$$\begin{aligned}
j^\mu &= \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \delta_S \phi_a - K^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} (-\varepsilon^\nu \partial_\nu \phi_a) + \varepsilon^\mu \mathcal{L} \\
&= -\varepsilon^\nu \underbrace{\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \partial_\nu \phi_a - \delta_\nu^\mu \mathcal{L} \right)}_{T^\mu{}_\nu} .
\end{aligned} \tag{115}$$

$T^\mu{}_\nu$: energy-momentum tensor

$$T^\mu{}_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \partial_\nu \phi_a - \delta_\nu^\mu \mathcal{L} \quad \longrightarrow \quad T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \partial^\nu \phi_a - \eta^{\mu\nu} \mathcal{L}. \tag{116}$$

This satisfies $\partial_\mu T^\mu{}_\nu = 0$, or $\partial_\mu T^{\mu\nu} = 0$.

3.4 Lecture 12: The Klein-Gordon and Dirac Equations

The Klein-Gordon equation¹ is

$$(\square + m^2)\phi = 0. \tag{117}$$

Then $\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x_i^2} + m^2 \phi = 0$, i.e. $E^2 = p^2 + m^2$.

The continuity equation gives us

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0, \quad \text{where} \quad \rho = i \left(\frac{\partial \phi^\dagger}{\partial t} \phi - \phi^\dagger \frac{\partial \phi}{\partial t} \right). \tag{118}$$

The Schrödinger equation is given by $i\partial_t \psi = -\nabla^2 \psi + V\psi$. We need to look for a relativistically covariant equation, that is, first order in time: $\hat{E} \rightarrow i\frac{\partial}{\partial t}$, $\hat{p} \rightarrow -i\nabla$.

$$i\partial_t \psi = H(\hat{p}, m)\psi = -i\alpha \cdot \nabla \psi + \beta m \psi, \quad \alpha, \beta \text{ constants.} \tag{119}$$

$$\begin{aligned}
HH\psi &= i\frac{\partial}{\partial t} \left(i\frac{\partial \psi}{\partial t} \right) \implies -\frac{\partial^2 \psi}{\partial t^2} = -(\alpha \cdot \nabla)^2 \psi + i(\alpha \cdot \nabla)\beta m \psi \\
&\quad + i(\beta m)(\alpha \cdot \nabla)\psi + \beta^2 m^2 \psi.
\end{aligned} \tag{120}$$

This must reproduce the KG equation. To satisfy all of these constraints, α, β cannot be scalars: they are matrices, and even ψ is a matrix. We have the following algebra for α, β :

Algebra of α_i, β

$$\beta^2 = \mathbb{K}, \quad \alpha_i^2 = \mathbb{K}, \quad \alpha_i \alpha_j + \alpha_j \alpha_i = 2\delta_{ij} \mathbb{K}, \quad \alpha_i \beta + \beta \alpha_i = 0. \tag{121}$$

This is because $(\alpha \cdot \nabla)^2 = \nabla^2 \implies \alpha_i^2 = \mathbb{K}$; $\beta^2 m^2 \psi = \beta^2 m \psi \implies \beta^2 = \mathbb{K}$; and $(\alpha \cdot \nabla)\beta m + \beta m(\alpha \cdot \nabla) = 0$.

Now $\alpha_i \beta = -\beta \alpha_i \implies \alpha_i \beta^2 = -\beta \alpha_i \beta \xrightarrow{\text{Tr}} \text{Tr}(\alpha_i) = \text{Tr}(-\beta \alpha_i \beta) \xrightarrow{\text{cyclic}} \text{Tr}(\alpha_i) = \text{Tr}(-\alpha_i) \implies \text{Tr}(\alpha_i) = 0$.

$$\alpha_i \beta \alpha_i = -\beta \alpha_i^2 \xrightarrow{\text{Tr}} \text{Tr}(\alpha_i \beta \alpha_i) = \text{Tr}(-\beta) \implies \text{Tr}(\alpha \beta) = \text{Tr}(-\beta) \implies \text{Tr}(\beta) = 0.$$

Both α_i and β are traceless.

Since $\beta^2 = \mathbb{K} \implies$ eigenvalues are $+1, -1$, but tracelessness $\implies$ the dimension is even. Similarly $\alpha_i^2 = \mathbb{K} \implies$ eigenvalues are $+1, -1$, but traceless $\implies$ dimension is even. In 2 dimensions we cannot

¹ $\square \equiv \partial^\mu \partial_\mu$.

have 3+1 anticommuting matrices, but in a 2D system we can have 2-dimensional β, α_i (only 3 anticommuting matrices are needed there). So β, α_i are 4-dimensional matrices at the least,

$$\beta = \begin{pmatrix} \mathbb{K}_2 & 0 \\ 0 & -\mathbb{K}_2 \end{pmatrix}, \quad \alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}. \quad (122)$$

$$i\frac{\partial\psi}{\partial t} + i\alpha \cdot \nabla\psi - \beta m\psi(t, \mathbf{x}) = 0, \quad \psi(t, \mathbf{x}) = \int \frac{d^4p}{(2\pi)^4} e^{-ipx} \tilde{\psi}(p). \quad (123)$$

$$\int \frac{d^4p}{(2\pi)^4} (p_0 - \alpha \cdot \mathbf{p} - m\beta) \tilde{\psi}(p) e^{-ipx} = 0 \quad \implies \quad (p_0 - \alpha \cdot \mathbf{p} - m\beta) \tilde{\psi} = 0. \quad (124)$$

$$\begin{pmatrix} (p_0 - m)\mathbb{K} & -\sigma \cdot \mathbf{p} \\ -\sigma \cdot \mathbf{p} & (p_0 + m)\mathbb{K} \end{pmatrix} \tilde{\psi}(p) = 0. \quad \text{But what is } p_0? \quad (125)$$

$(A - \lambda\mathbb{K})\psi = 0 \Rightarrow \det(A - \lambda\mathbb{K}) = 0$, so

$$(p_0^2 - p^2 - m^2)^2 = 0 \quad \implies \quad p_0 = \pm\sqrt{p^2 + m^2}. \quad (126)$$

Let $\psi(p) = \begin{pmatrix} \phi(p) \\ \chi(p) \end{pmatrix}$, giving

$$(p_0 - m)\phi - (\mathbf{p} \cdot \sigma)\chi = 0, \quad -(\mathbf{p} \cdot \sigma)\phi + (p_0 + m)\chi = 0. \quad (127)$$

For $p_0 > 0 \Rightarrow p_0 \geq m$, and $\chi(p) = \frac{(\mathbf{p} \cdot \sigma)\phi(p)}{p_0 + m}$, so

$$(p_0^2 - m^2)\phi - \frac{(\mathbf{p} \cdot \sigma)^2}{p_0 + m}\phi = 0 \quad \implies \quad (p_0^2 - m^2 - |\mathbf{p}|^2)\phi = 0 \quad \implies \quad |\mathbf{p}|^2 = p_0^2 - m^2 \geq 0. \quad (128)$$

Choosing $\phi(p) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ gives two solutions for $p_0 > 0$,

$$\psi(p) = \begin{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \frac{\sigma \cdot \mathbf{p}}{p_0 + m} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix}, \quad \begin{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \frac{\sigma \cdot \mathbf{p}}{p_0 + m} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix}. \quad (129)$$

For $p_0 < 0, p_0 = -E_p, \Rightarrow \phi = -\frac{(\sigma \cdot \mathbf{p})\chi}{E_p + m}$, with $\chi = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, giving

$$\psi = \begin{pmatrix} -\frac{\sigma \cdot \mathbf{p}}{E_p + m} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix}, \quad \begin{pmatrix} -\frac{\sigma \cdot \mathbf{p}}{E_p + m} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix}. \quad (130)$$

3.5 Lecture 13: Solutions of the Dirac Equation

The Dirac equation is given by $(p_0 - \alpha \cdot \mathbf{p} - \beta m)\psi(p) = 0$. We get 4 solutions, with $E_p = \sqrt{|\mathbf{p}|^2 + m^2}$:

$$p_0 > 0, p_0 = E_p : \quad u_1 = \begin{pmatrix} 1 \\ 0 \\ p_3/(E_p + m) \\ (p_1 + ip_2)/(E_p + m) \end{pmatrix}, \quad u_2 = \begin{pmatrix} 0 \\ 1 \\ (p_1 - ip_2)/(E_p + m) \\ -p_3/(E_p + m) \end{pmatrix}, \quad (131)$$

$$p_0 < 0, p_0 = -E_p : \quad v_1 = \begin{pmatrix} -p_3/(E_p + m) \\ -(p_1 + ip_2)/(E_p + m) \\ 1 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} -(p_1 - ip_2)/(E_p + m) \\ p_3/(E_p + m) \\ 0 \\ 1 \end{pmatrix}. \quad (132)$$

Since $\hat{H} = \alpha \cdot \mathbf{p} + \beta m \Rightarrow [\hat{H}, \mathbf{p}] = 0$. Consider an operator

$$\Sigma = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix}, \quad \alpha_j = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}. \quad (133)$$

$$\begin{aligned} [\Sigma_i, \alpha_j] &= \begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix} \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix} - \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix} \begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix} \\ &= \begin{pmatrix} 0 & \sigma_i \sigma_j \\ \sigma_i \sigma_j & 0 \end{pmatrix} - \begin{pmatrix} 0 & \sigma_j \sigma_i \\ \sigma_j \sigma_i & 0 \end{pmatrix} = \begin{pmatrix} 0 & [\sigma_i, \sigma_j] \\ [\sigma_i, \sigma_j] & 0 \end{pmatrix} \\ &= i\varepsilon_{ijk} \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix}. \end{aligned} \quad (134)$$

$$\begin{aligned} [\Sigma_i, \beta] &= \begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix} \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} - \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix} \\ &= \begin{pmatrix} \sigma_i & 0 \\ 0 & -\sigma_i \end{pmatrix} - \begin{pmatrix} \sigma_i & 0 \\ 0 & -\sigma_i \end{pmatrix} = 0. \end{aligned} \quad (135)$$

Hence $[\Sigma_i, \beta] = 0 \Rightarrow [\Sigma_i, \hat{H}] = i\varepsilon_{ijk} \alpha_k \hat{p}_j = i(\alpha \times \mathbf{p})_i$.

Consider a helicity operator, $\hat{h} = \frac{1}{2} \frac{\Sigma \cdot \mathbf{p}}{|\mathbf{p}|} \Rightarrow [\hat{h}, \hat{H}] = i \frac{(\alpha \times \mathbf{p}) \cdot \mathbf{p}}{|\mathbf{p}|} = 0$. So we can define a CSCO as $\{H, \mathbf{p}, \hat{h}\}$: the projection of spin along the direction of momentum is conserved.

In the rest frame, taking $\hat{h}, \mathbf{p}$ along z and the $|\mathbf{p}| \rightarrow 0$ limit,

$$\hat{h} = \frac{1}{2} \Sigma_z = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad (136)$$

and the solutions of the Dirac equation become

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad u_1 \quad u_2 \quad v_1 \quad v_2. \quad (137)$$

$$\hat{h}u_1 = \frac{1}{2}u_1, \quad \hat{h}u_2 = -\frac{1}{2}u_2 \quad (p_0 = E_p); \quad \hat{h}v_1 = \frac{1}{2}v_1, \quad \hat{h}v_2 = -\frac{1}{2}v_2 \quad (p_0 = -E_p). \quad (138)$$

The Dirac equation in Lorentz covariant form: multiplying $(i\partial_0 + i\alpha \cdot \nabla - \beta m)\psi = 0$ through by $i\beta$,

$$(i\beta\partial_0 + i\beta\alpha^i\partial_i - m)\psi = 0. \quad (139)$$

Define $\gamma^0 = \beta$, $\gamma^i = \beta\alpha^i \Rightarrow (i\gamma^\mu\partial_\mu - m)\psi = 0$.

Since $(\alpha_i)^\dagger = \alpha_i$, $\beta^\dagger = \beta \Rightarrow (\gamma^0)^\dagger = \gamma^0$, $(\gamma^i)^\dagger = -\gamma^i$. Since $(\alpha_i)^2 = \mathbb{K} = \beta^2 \Rightarrow (\gamma^0)^2 = \mathbb{K}$, $(\gamma^i)^2 = -\mathbb{K}$.

$$\{\gamma^0, \gamma^i\} = \gamma^0\gamma^i + \gamma^i\gamma^0 = \beta^2\alpha_i + \beta\alpha_i\beta = \alpha_i + (-\alpha_i) = 0, \quad (140)$$

$$\{\gamma^i, \gamma^j\} = \beta\alpha_i\beta\alpha_j + \beta\alpha_j\beta\alpha_i = -(\alpha_i\alpha_j + \alpha_j\alpha_i) = -\{\alpha_i, \alpha_j\}. \quad (141)$$

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbb{K}. \quad (142)$$

3.6 Lecture 14: Clifford Algebra

$\text{tr}(\gamma^\mu) = 0$: for a 2D QM system, the CSCO is $\{\mathbb{K}, \sigma_i\}$, $i = 1, 2, 3$. Using $\{\sigma_i, \sigma_j\} = 2\delta_{ij}\mathbb{K}$ and $[\sigma_i, \sigma_j] = 2i\varepsilon_{ijk}\sigma_k$,

$$\sigma_i\sigma_j = \mathbb{K}\delta_{ij} + i\varepsilon_{ijk}\sigma_k, \quad (\gamma^0)^\dagger = \gamma^0, \quad (\gamma^i)^\dagger = -\gamma^i. \quad (143)$$

Similarly $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbb{K}$. The Clifford algebra generated by the γ^μ has $2^4 = 16$ basis elements, of definite rank:

- $\mathbb{K} \rightarrow \text{rank } 0$,
- $\gamma^\mu \rightarrow \text{rank } 1$,
- $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu] \rightarrow \text{rank } 2$,
- $\gamma^5\gamma^\mu \rightarrow \text{rank } 3$,
- $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \rightarrow \text{rank } 4$ (the maximal-length string, which cannot be shrunk).

There are 16 such algebraic structures. From

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}\mathbb{K} \quad (144)$$

$$\text{Tr}(\gamma^\mu\gamma^\nu) + \text{Tr}(\gamma^\nu\gamma^\mu) = 2\eta^{\mu\nu}\text{Tr}(\mathbb{K}) \quad \implies \quad \text{Tr}(\gamma^\mu\gamma^\nu) = 4\eta^{\mu\nu}. \quad (145)$$

Clifford Algebra Elements

- Scalar: $\mathbb{K}$, $\text{tr}(\mathbb{K}) = 4$.
- Vector: γ^μ , $\text{tr}(\gamma^\mu) = 0$.
- Tensor: $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$, $\text{tr}(\sigma^{\mu\nu}) = 0$.
- Axial vector: $\gamma^\mu\gamma^5$, $\text{tr}(\gamma^\mu\gamma^5) = 0$.
- Pseudoscalar: $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$, $\text{tr}(\gamma^5) = 0$.

$$\begin{aligned} (\gamma^5)^2 &= i^2\gamma^0\gamma^1\gamma^2\gamma^3\gamma^0\gamma^1\gamma^2\gamma^3 = (-1)(-(\gamma^0)^2)(\gamma^1)^2(-(\gamma^2)^2)(-(\gamma^3)^2) \\ &= \mathbb{K}. \end{aligned} \quad (146)$$

$$\begin{aligned}
(\gamma^5)^\dagger &= (-i)(\gamma^3)^\dagger(\gamma^2)^\dagger(\gamma^1)^\dagger(\gamma^0)^\dagger = (-i)(-\gamma^3)(-\gamma^2)(-\gamma^1)\gamma^0 \\
&= (-i)\gamma^3\gamma^2\gamma^1\gamma^0 \\
&= i\gamma^2\gamma^1\gamma^0\gamma^3(-1) \\
&= i\gamma^1\gamma^0\gamma^2\gamma^3(-1) \\
&= i\gamma^0\gamma^1\gamma^2\gamma^3 = \gamma^5.
\end{aligned} \tag{147}$$

$$\{\gamma^5, \gamma^\mu\} = \gamma^5\gamma^\mu + \gamma^\mu\gamma^5 = 0 \quad (\text{check for each } \mu). \tag{148}$$

Using $\langle e^i | e^j \rangle = \text{tr}(e^i e^j)$ as an inner product on this 16-dimensional space of matrices, the basis elements are pairwise orthogonal, e.g.

$$\begin{aligned}
\langle \mathbb{K} | \gamma^\mu \rangle &= \text{tr}(\gamma^\mu) = 0, & \langle \mathbb{K} | \sigma^{\mu\nu} \rangle &= \text{tr}(\sigma^{\mu\nu}) = 0, \\
\langle \mathbb{K} | \gamma^\mu \gamma^5 \rangle &= \text{tr}(\gamma^\mu \gamma^5) = 0, & \langle \mathbb{K} | \gamma^5 \rangle &= \text{tr}(\gamma^5) = 0, \\
\langle \gamma^\mu | \sigma^{\alpha\beta} \rangle &= \text{tr}(\gamma^\mu \sigma^{\alpha\beta}) = 0, & \langle \gamma^\mu | \gamma^5 \gamma^\nu \rangle &= \text{tr}(\gamma^\mu \gamma^5 \gamma^\nu) = 0, \\
\langle \sigma^{\mu\nu} | \gamma^5 \gamma^\alpha \rangle &= \text{tr}(\sigma^{\mu\nu} \gamma^5 \gamma^\alpha) = 0, & \langle \sigma^{\mu\nu} | \gamma^5 \rangle &= \text{tr}(\gamma^5 \sigma^{\mu\nu}) = 0, \\
\langle \gamma^5 \gamma^\mu | \gamma^5 \rangle &= \text{tr}(\gamma^\mu) = 0, & \langle \sigma^{\mu\nu} | \sigma^{\alpha\beta} \rangle &= \eta(\dots), \\
\langle \gamma^5 \gamma^\mu | \gamma^5 \gamma^\nu \rangle &= \text{tr}(\gamma^5 \gamma^\mu \gamma^5 \gamma^\nu) = 4\eta^{\mu\nu} \mathbb{K}.
\end{aligned} \tag{149}$$

Consider $\gamma^\mu \gamma^\nu \gamma^\alpha \gamma^\beta = -\gamma^\nu \gamma^\mu \gamma^\alpha \gamma^\beta + 2\eta^{\mu\nu} \gamma^\alpha \gamma^\beta$. We know $\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\eta^{\mu\nu} \mathbb{K}$, and $\text{tr}(\gamma^\mu \gamma^\nu) = 4\eta^{\mu\nu}$. Then

$$\begin{aligned}
\gamma^\mu \gamma^\nu \gamma^\alpha \gamma^\beta &= \gamma^\nu \gamma^\alpha \times 2\eta^{\mu\beta} - \gamma^\nu \gamma^\beta \gamma^\alpha \gamma^\mu + 2\eta^{\mu\nu} \gamma^\alpha \gamma^\beta \\
&= -\gamma^\nu \gamma^\alpha \gamma^\beta \gamma^\mu + 2\eta^{\mu\beta} \gamma^\nu \gamma^\alpha + 2\eta^{\mu\nu} \gamma^\alpha \gamma^\beta - 2\eta^{\mu\alpha} \gamma^\nu \gamma^\beta,
\end{aligned} \tag{150}$$

$$\gamma^\mu \gamma^\nu \gamma^\alpha \gamma^\beta + \gamma^\nu \gamma^\alpha \gamma^\beta \gamma^\mu = 2 \left(\eta^{\mu\beta} \gamma^\nu \gamma^\alpha + \eta^{\mu\nu} \gamma^\alpha \gamma^\beta - \eta^{\mu\alpha} \gamma^\nu \gamma^\beta \right). \tag{151}$$

Taking the trace (which is the same on both sides, by the cyclic property),

$$2 \text{Tr}(\gamma^\mu \gamma^\nu \gamma^\alpha \gamma^\beta) = 8 \left(\eta^{\mu\nu} \eta^{\alpha\beta} - \eta^{\mu\alpha} \eta^{\nu\beta} + \eta^{\mu\beta} \eta^{\nu\alpha} \right), \tag{152}$$

$$\text{Tr}(\gamma^\mu \gamma^\nu \gamma^\alpha \gamma^\beta) = 4 \left(\eta^{\mu\nu} \eta^{\alpha\beta} - \eta^{\mu\alpha} \eta^{\nu\beta} + \eta^{\mu\beta} \eta^{\nu\alpha} \right). \tag{153}$$

Consider

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \frac{i}{4!} \varepsilon_{\mu\nu\alpha\beta} \gamma^\mu \gamma^\nu \gamma^\alpha \gamma^\beta, \quad \varepsilon_{0123} = +1, \quad \varepsilon^{0123} = -1. \tag{154}$$

3.7 Class 15: Lorentz Covariance of the Dirac Equation

$(\gamma^0)^\dagger = \gamma^0$ and $(\gamma^i)^\dagger = -\gamma^i$. So $(\gamma^\mu)^\dagger = \gamma^0 \gamma^\mu \gamma^0$.

$$\begin{aligned}
(\sigma^{\mu\nu})^\dagger &= \left(\frac{i}{2} (\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu) \right)^\dagger = -\frac{i}{2} \left((\gamma^\nu)^\dagger (\gamma^\mu)^\dagger - (\gamma^\mu)^\dagger (\gamma^\nu)^\dagger \right) \\
&= -\frac{i}{2} \gamma^0 (\gamma^\nu \gamma^\mu - \gamma^\mu \gamma^\nu) \gamma^0 = \gamma^0 \sigma^{\mu\nu} \gamma^0.
\end{aligned} \tag{155}$$

$$\begin{aligned}
(\gamma^5 \gamma^\mu)^\dagger &= (\gamma^\mu)^\dagger (\gamma^5)^\dagger = \gamma^0 \gamma^\mu \gamma^0 \gamma^5 = -\gamma^0 \gamma^\mu \gamma^5 \gamma^0 \\
&= \gamma^0 (\gamma^5 \gamma^\mu) \gamma^0.
\end{aligned} \tag{156}$$

Lorentz covariance of the Dirac equation

$(i\gamma^\mu \partial_\mu - m)\psi(x) = 0$ under a Lorentz transform $x^\mu \rightarrow \Lambda^\mu{}_\nu x^\nu$. Let $\psi(x) \rightarrow \psi'(\Lambda^\mu{}_\nu x^\nu) = S(\Lambda)\psi(x) \Rightarrow \psi(x) = S^{-1}(\Lambda)\psi'(x')$.

$$(i\gamma^\mu \Lambda^\nu{}_\mu \partial'_\nu - m) S^{-1}(\Lambda)\psi'(x') = 0, \quad (157)$$

$$S(\Lambda) (i\gamma^\mu \Lambda^\nu{}_\mu \partial'_\nu - m) S^{-1}(\Lambda) - m\psi'(x') = 0. \quad (158)$$

If $i\gamma^\mu S(\Lambda)\Lambda^\nu{}_\mu \partial'_\nu S^{-1}(\Lambda) = i\gamma^\mu \partial_\mu$ then the transformed equation is again the Dirac equation, i.e.

$$S(\Lambda)\gamma^\mu \Lambda^\nu{}_\mu S^{-1}(\Lambda) = \gamma^\nu \quad \implies \quad S^{-1}\gamma^\nu S = \gamma^\mu \Lambda^\nu{}_\mu. \quad (159)$$

Let $\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu$; then

$$S(\Lambda) = \exp \left[-\frac{i}{4} \omega_{\mu\nu} \sigma^{\mu\nu} \right]. \quad (160)$$

$\psi \rightarrow \psi' = S\psi$, and $\psi^\dagger \psi \rightarrow \psi^\dagger S^\dagger S \psi \neq \psi^\dagger \psi \Rightarrow S^\dagger S$ is *not* the identity, i.e. S is *not* a unitary matrix.

$$S^\dagger(\Lambda) = \exp \left[\frac{i}{4} \omega_{\mu\nu} (\sigma^{\mu\nu})^\dagger \right] = \gamma^0 S^{-1}(\Lambda) \gamma^0. \quad (161)$$

Define $\bar{\psi} \equiv \psi^\dagger \gamma^0$. Then

$$(\bar{\psi})' = (S\psi)^\dagger \gamma^0 = \psi^\dagger S^\dagger \gamma^0 = \bar{\psi} S^{-1} \quad (162)$$

$$\implies \quad \bar{\psi} \psi \rightarrow \bar{\psi} S^{-1} S \psi = \bar{\psi} \psi. \quad \text{So } \bar{\psi} = \psi^\dagger \gamma^0 \text{ is unitary.} \quad (163)$$

Bilinear covariants

$$\text{Scalar: } \bar{\psi} \not{K} \psi, \quad (164)$$

$$\text{Vector: } \bar{\psi}_1 \gamma^\mu \psi_2 \rightarrow \bar{\psi} S^{-1} \gamma^\mu S \psi = \bar{\psi} \gamma^\nu \Lambda^\mu{}_\nu \psi = (\bar{\psi} \gamma^\nu \psi) \Lambda^\mu{}_\nu, \quad (165)$$

$$\text{Tensor: } \bar{\psi} \sigma^{\mu\nu} \psi_2 \rightarrow \bar{\psi} S^{-1} \sigma^{\mu\nu} S \psi = (\bar{\psi} \sigma^{\alpha\beta} \psi) \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta. \quad (166)$$

$$\text{Pseudoscalar: } \bar{\psi} \gamma^5 \psi \Rightarrow \bar{\psi} S^{-1} \gamma^5 S \psi.$$

$$\begin{aligned} \bar{\psi} S^{-1} \left[\frac{i}{4!} \varepsilon_{\mu\nu\alpha\beta} \gamma^\mu \gamma^\nu \gamma^\alpha \gamma^\beta \right] S \psi &= \frac{i}{4!} \varepsilon_{\mu\nu\alpha\beta} \bar{\psi} S^{-1} \gamma^\mu S S^{-1} \gamma^\nu S S^{-1} \gamma^\alpha S S^{-1} \gamma^\beta S \psi \\ &= \frac{i}{4!} \varepsilon_{\mu\nu\alpha\beta} \Lambda^\mu{}_{\mu'} \Lambda^\nu{}_{\nu'} \Lambda^\alpha{}_{\alpha'} \Lambda^\beta{}_{\beta'} \bar{\psi} \gamma^{\mu'} \gamma^{\nu'} \gamma^{\alpha'} \gamma^{\beta'} \psi \\ &= (\det \Lambda) \frac{i}{4!} \varepsilon_{\mu'\nu'\alpha'\beta'} \bar{\psi} \gamma^{\mu'} \gamma^{\nu'} \gamma^{\alpha'} \gamma^{\beta'} \psi \\ &= \det(\Lambda) \gamma^5. \end{aligned} \quad (167)$$

Axial vector:

$$\det(\Lambda) (\bar{\psi} \gamma^5 \gamma^\nu \psi) \Lambda^\mu{}_\nu \longleftarrow \bar{\psi} \gamma^5 \gamma^\mu \psi. \quad (168)$$

3.8 Tutorial 1: Group and Representation Theory

A group can have many representations.

Representation

A **representation** is a map $D : G \rightarrow O$ (some set of operators) such that

$$D(e) = \mathbb{K}, \quad D(g_1)D(g_2) = D(g_1 g_2). \quad (169)$$

For C_3 (the cyclic group of order 3): $D(e) = 1$, $D(g) = e^{2\pi i/3}$, $D(g^2) = e^{4\pi i/3}$ is one representation of C_3 . Another is

$$D(e) = \mathbb{I}_3, \quad D(g) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad D(g^2) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}. \quad (170)$$

$GL(n, F)$ is the set of invertible matrices, where n is the dimension and F is the field.

Irreducible representations

Let D be a representation, v a vector, $v \in V$. If

$$D(g)v \in H \quad \forall v \in H, \forall g \in G, \quad (171)$$

and H is an invariant subspace, then, if such an H exists, the representation is *reducible*; otherwise it is *irreducible*.

In QM, we have a $2 \otimes 2$ spin- $\frac{1}{2}$ system, $(|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle)$. We can also represent the $SU(2)$ group as

$$\left\{ |\uparrow\uparrow\rangle, |\downarrow\downarrow\rangle, \frac{|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle}{\sqrt{2}} \right\} \quad (3\text{-dim subspace}), \quad \frac{|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle}{\sqrt{2}} \quad (1\text{-dim subspace}). \quad (172)$$

The first representation is reducible; the second is irreducible.

Translations and the Lie algebra

Consider translations $x \rightarrow x + a$, and let T be an operator. Let $T(a)f(x) = f(x + a) = f(x) + af'(x) + \frac{a^2}{2}f''(x) + \dots$ (Taylor series)

$$= f(x) \left(1 + a\partial_x + \frac{a^2}{2}\partial_x^2 + \dots \right) = e^{a\partial_x} f(x). \quad (173)$$

Continuous operations are infinitely many infinitesimal translations stacked up on each other. The infinitesimal operator is $\left(1 + \delta a \frac{\partial}{\partial x} \right)$; no. of parameters = no. of generators.

$$\left(1 + \delta a \frac{\partial}{\partial x} \right)^N = 1 + N\delta a \frac{\partial}{\partial x} + \frac{(N\delta a)^2}{2} \frac{\partial^2}{\partial x^2} + \dots = e^{a\partial_x} \quad (a = N\delta a). \quad (174)$$

Two translations commute: $T(a_x)T(a_y) = T(a_y)T(a_x) \Rightarrow [p_x, p_y] = 0$.

Definition: Lie Algebra

A **Lie algebra** $\mathfrak{g}$ is a vector space of generators with an operation $[\cdot, \cdot]$ defined over them.

For reducible representations, there exists a similarity transform that brings the representation to block-diagonal form,

$$S^{-1}D(g)S = \begin{pmatrix} D_1(g) & & \\ & D_2(g) & \\ & & \ddots \end{pmatrix} \quad \forall D(g), \quad D(g) = D_1(g) \oplus D_2(g) \oplus \dots, \quad (175)$$

where $D(g)$ is a direct sum of $D_i(g)$. If no such S exists, then the representation is *irreducible*.

The commutator satisfies

$$[x, y] = -[y, x], \quad [x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0 \quad (\text{Jacobi identity}). \quad (176)$$

The rotation group in 3D

$SO(3)$ has 3 generators; $R^\top R = \mathbb{I}$ gives 6 constraints.

$$R_z = \begin{pmatrix} \cos \delta\psi & -\sin \delta\psi & 0 \\ \sin \delta\psi & \cos \delta\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{\delta\psi \text{ small}} \mathbb{I} + \underbrace{\delta\psi \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{\text{generator}}, \quad (177)$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \rightarrow \begin{pmatrix} x - \delta\psi y \\ y + \delta\psi x \\ z \end{pmatrix} \implies \delta \mathbf{r} = \delta\psi_z \times \mathbf{r}. \quad (178)$$

$$[L_x, L_y] = i\hbar L_z \implies SO(3) \text{ algebra.} \quad (179)$$

Definition: Lie Group

A group $(G, \circ)$ is a **Lie group** if

- (i) G is a smooth manifold (the *group manifold*); we can define tangents to group manifolds – the number of tangent vectors on this manifold is called a *generator*.
- (ii) $G \times G \rightarrow G$, $\cdot^{-1} : G \rightarrow G$, are smooth (C^∞) maps.

Any given $g(\alpha) = e + \alpha^a \partial_{x^a} g + \dots$, where α parametrises the generators. Any vector space which follows

$$[\ , \] = \partial_{x^a} \partial_{x^b} - \partial_{x^b} \partial_{x^a} \quad (180)$$

is the Lie algebra, which is always “near” the identity: a local property. For $\det(\Lambda) = -1$ there is no continuous path to the identity, and so it cannot be studied by Lie algebras. Hence why we study $SO^+(3, 1)$. $SU(2)$ is a double cover of $SO^+(3, 1)$, since these groups are, by definition, matrix groups.

$$SU(2) = \left\{ 2 \times 2 \ M_n(\mathbb{C}) \mid U^\dagger U = \mathbb{I} \right\} \ \& \ \det(U) = 1. \quad (181)$$

Consider $U(t) = \mathbb{I} + tT$, t the parameter, T the generator. For $U^\dagger U = \mathbb{I} \implies T^\dagger = -T$: antihermitian. $\det(e^{tT}) = e^{t \text{Tr}(T)} = 1 \implies \text{Tr}(T) = 0 \implies$ generators are also traceless.

$$\begin{pmatrix} a & b + ic \\ b + ic & -a \end{pmatrix} = T \longrightarrow a\sigma_z + b\sigma_x + c\sigma_y, \quad T_i = \frac{\sigma_i}{2}, \quad (182)$$

so $\sigma_x, \sigma_y, \sigma_z$ are a basis, $\dim(SU(2)) = 3$, and $[T_i, T_j] = i\varepsilon_{ijk} T_k$.

Casimir Operator

An operator C that commutes with all generators: $[C, T_i] = 0$.

For angular momentum, $J^2 = J_1^2 + J_2^2 + J_3^2$. We can make a CSCO $\{J_3, J^2\}$. Define $J_\pm = J_x \pm iJ_y$; $J_+ |\lambda, m\rangle \geq 0$, $J_- |\lambda, m-1\rangle \geq 0$: $2j+1$ states, $j = N/2$.

Consider $U(1) = \{e^{i\phi} \mid \phi \in [0, 2\pi)\}$; $e^{iq\phi}$, $q \in \mathbb{Z}$, is also a representation.

The Lorentz algebra, revisited

For $O(1, 3)$: $\Lambda^\top \eta \Lambda = \eta \implies \det(\Lambda) = \pm 1 \implies SO(1, 3)$; $\eta_{00} \geq 1 \rightarrow$ important for causality: $SO_+(1, 3)$.

$$\Lambda^\mu{}_\nu = \delta^\mu_\nu + \omega^\mu{}_\nu \implies \omega_{\mu\nu} = -\omega_{\nu\mu}, \quad \exp\left(-\frac{i}{2} \omega_{\alpha\beta} M^{\alpha\beta}\right). \quad (183)$$

$$M^{\mu\nu} = i(x^\mu \partial^\nu - x^\nu \partial^\mu). \quad (184)$$

Let $K^i = M^{0i}$, $J^i = \frac{1}{2}\varepsilon^{ijk}M^{jk} - 6$ generators: 3 boosts, 3 rotations.

$$[J^i, J^j] = i\varepsilon^{ijk}J^k, \quad [K^i, K^j] = -i\varepsilon^{ijk}J^k, \quad [J^i, K^j] = i\varepsilon^{ijk}K^k. \quad (185)$$

Define $J_\pm^i = \frac{1}{2}(J^i \pm K^i)$. Then

$$[J_+^i, J_+^j] = i\varepsilon^{ijk}J_+^k, \quad [J_-^i, J_-^j] = i\varepsilon^{ijk}J_-^k, \quad [J_+^i, J_-^j] = 0. \quad (186)$$

$$SO(1, 3) \cong SU(2)_L \otimes SU(2)_R. \quad (187)$$

$(\frac{1}{2}, 0) \oplus (0, \frac{1}{2}) \rightarrow$ Dirac spinor: $(\frac{1}{2}, 0) \rightarrow$ left-handed bifermion, $(0, \frac{1}{2}) \rightarrow$ right-handed bifermion.
 $(\frac{1}{2}, \frac{1}{2})$: $2 \otimes 2 = 0 \oplus 1$.

$$J^3 = xp_y - yp_x \xrightarrow{P} J^3, \quad PJ_\pm P^{-1} = \mathbf{J}_\mp, \quad \begin{pmatrix} \psi_{L+} \\ \psi_{R-} \end{pmatrix} \xrightarrow{P} \begin{pmatrix} \psi_{R+} \\ \psi_{L-} \end{pmatrix}. \quad (188)$$

3.9 Lecture 16: Contracting Gamma Matrices

$$\begin{aligned} \gamma^\mu \gamma_\mu &= \eta^{\mu\nu} \gamma_\nu \gamma_\mu = \eta^{\mu\nu} [2\eta_{\mu\nu} \mathbb{K} - \gamma_\mu \gamma_\nu] = 2(4 \otimes \mathbb{K}) - \gamma_\mu \gamma^\mu \\ &\implies \boxed{\gamma^\mu \gamma_\mu = 4\mathbb{K}}. \end{aligned} \quad (189)$$

$$\begin{aligned} \gamma^\mu \gamma^\nu \gamma_\mu &= (2\eta^{\mu\nu} \mathbb{K} - \gamma^\nu \gamma^\mu) \gamma_\mu = 2\eta^{\mu\nu} \gamma_\mu - \underbrace{\gamma^\nu \gamma^\mu \gamma_\mu}_{4\mathbb{K}} \\ &= 2\eta^{\mu\nu} \gamma_\mu - 4\gamma^\nu \\ &\implies \boxed{\gamma^\mu \gamma^\nu \gamma_\mu = -2\gamma^\nu}. \end{aligned} \quad (190)$$

$$\begin{aligned} \gamma^\mu \gamma_\nu \gamma_\rho \gamma_\mu &= \gamma^\mu \gamma_\nu [2\eta_{\rho\mu} - \gamma_\mu \gamma_\rho] \\ &= 2\gamma_\rho \gamma_\nu - (\gamma^\mu \gamma_\nu \gamma_\mu) \gamma_\rho \\ &= 2\gamma_\rho \gamma_\nu + 2\gamma_\nu \gamma_\rho = 2\{\gamma_\rho, \gamma_\nu\} = 4\eta_{\rho\nu} \mathbb{K}. \end{aligned} \quad (191)$$

$$\begin{aligned} \gamma^\mu \gamma_\nu \gamma_\rho \gamma_\sigma \gamma_\mu &= \gamma^\mu \gamma_\nu \gamma_\rho (2\eta_{\sigma\mu} - \gamma_\mu \gamma_\sigma) \\ &= 2\gamma_\sigma \gamma_\nu \gamma_\rho - (\gamma^\mu \gamma_\nu \gamma_\rho \gamma_\mu) \gamma_\sigma \\ &= 2\gamma_\sigma \gamma_\nu \gamma_\rho - 4\eta_{\nu\rho} \gamma_\sigma \\ &= 2\gamma_\sigma \gamma_\nu \gamma_\rho - 2\gamma_\sigma (\gamma_\nu \gamma_\rho + \gamma_\rho \gamma_\nu) \\ &= -2\gamma_\sigma \gamma_\rho \gamma_\nu \\ &\implies \boxed{\gamma^\mu \gamma_\nu \gamma_\rho \gamma_\sigma \gamma_\mu = -2\gamma_\sigma \gamma_\rho \gamma_\nu}. \end{aligned} \quad (192)$$

Outer product of spinors

$$\psi_2 \bar{\psi}_1 = a\mathbb{K} + b_\mu \gamma^\mu + c_{\mu\nu} \sigma^{\mu\nu} + d_\mu \gamma^\mu \gamma^5 + e \gamma^5. \quad (193)$$

Consider $\text{Tr}(\psi_2 \bar{\psi}_1) = 4a$ ($\text{Tr}(\mathbb{K}) = 4$ and the rest are traceless):

$$a = \frac{1}{4} \text{Tr}(\psi_2 \bar{\psi}_1) = \frac{1}{4} \bar{\psi}_1 \psi_2 \quad (\text{Tr}(\psi_2 \bar{\psi}_1) = (\psi_2)_i (\bar{\psi}_1)_i = (\bar{\psi}_1)_i (\psi_2)_i = \bar{\psi}_1 \psi_2). \quad (194)$$

$$\text{Tr}(\gamma^\nu \psi_2 \bar{\psi}_1) = \text{Tr}(b_\mu \gamma^\nu \gamma^\mu) = \text{Tr}(b_\mu 4\eta^{\mu\nu}) = 4b^\nu \implies b^\nu = \frac{1}{4} \text{Tr}(\gamma^\nu \psi_2 \bar{\psi}_1) = \frac{1}{4} \bar{\psi}_1 \gamma^\nu \psi_2. \quad (195)$$

$$\begin{aligned}\text{Tr}(\sigma^{\alpha\beta}\psi_2\bar{\psi}_1) &= c_{\mu\nu}\text{Tr}(\sigma^{\alpha\beta}\sigma^{\mu\nu}) = c_{\mu\nu}\text{Tr}\left(4\eta^{\mu\alpha}\eta^{\nu\beta} - \eta^{\mu\beta}\eta^{\nu\alpha}\right) \\ &= 4\left(c^{\alpha\beta} - c^{\beta\alpha}\right) = 8c^{\alpha\beta} \quad (c^{\alpha\beta} \text{ antisymmetric}),\end{aligned}\quad (196)$$

$$c^{\alpha\beta} = \frac{1}{8}\text{Tr}(\gamma^{\alpha\beta}\psi_2\bar{\psi}_1) = \frac{1}{8}\bar{\psi}_1\sigma^{\alpha\beta}\psi_2. \quad (197)$$

$$\text{Tr}(\gamma^\nu\gamma^5\psi_2\bar{\psi}_1) = \text{Tr}(d_\mu\gamma^\nu\gamma^5\gamma^\mu\gamma^5) = -d_\mu\text{Tr}(\gamma^\nu\gamma^\mu) = -d_\mu 4\eta^{\mu\nu} = -4d^\nu, \quad (198)$$

$$d^\nu = -\frac{1}{4}\bar{\psi}_1\gamma^\nu\gamma^5\psi_2. \quad (199)$$

$$\text{Tr}(\gamma^5\psi_2\bar{\psi}_1) = \text{Tr}(e\gamma^5\gamma^5) = 4e \quad \implies \quad \boxed{e = \frac{1}{4}\bar{\psi}_1\gamma^5\psi_2}. \quad (200)$$

$$\psi_2\bar{\psi}_1 = \frac{1}{4}\bar{\psi}_1\psi_2 + \frac{1}{4}(\bar{\psi}_1\gamma_\mu\psi_2)\gamma^\mu + \frac{1}{8}(\bar{\psi}_1\sigma_{\mu\nu}\psi_2)\sigma^{\mu\nu} - \frac{1}{4}(\bar{\psi}_1\gamma_\mu\gamma^5\psi_2)\gamma^\mu\gamma^5 + \frac{1}{4}(\bar{\psi}_1\gamma^5\psi_2)\gamma^5. \quad (201)$$

The Fierz identity

$$(\bar{\psi}_3\psi_2)(\bar{\psi}_1\psi_4) = \bar{\psi}_3[\]\psi_4 \quad (202)$$

where, inserting the outer-product expansion above for $\psi_2\bar{\psi}_1$ between $\bar{\psi}_3$ and ψ_4 ,

$$\begin{aligned}(\bar{\psi}_3\psi_2)(\bar{\psi}_1\psi_4) &= \frac{1}{4}(\bar{\psi}_1\psi_2)(\bar{\psi}_3\psi_4) + \frac{1}{4}(\bar{\psi}_1\gamma_\mu\psi_2)(\bar{\psi}_3\gamma^\mu\psi_4) + \frac{1}{8}(\bar{\psi}_1\sigma_{\mu\nu}\psi_2)(\bar{\psi}_3\sigma^{\mu\nu}\psi_4) \\ &\quad - \frac{1}{4}(\bar{\psi}_1\gamma_\mu\gamma^5\psi_2)(\bar{\psi}_3\gamma^\mu\gamma^5\psi_4) + \frac{1}{4}(\bar{\psi}_1\gamma^5\psi_2)(\bar{\psi}_3\gamma^5\psi_4).\end{aligned}\quad (203)$$

Defining the bilinears

$$\bar{\psi}_1\psi_2 = S_{12}, \quad \bar{\psi}_1\gamma^\mu\psi_2 = V_{12}^\mu, \quad \bar{\psi}_1\sigma^{\mu\nu}\psi_2 = T_{12}^{\mu\nu}, \quad \bar{\psi}_1\gamma^\mu\gamma^5\psi_2 = A_{12}^\mu, \quad \bar{\psi}_1\gamma^5\psi_2 = P_{12}, \quad (204)$$

$$S_{32}S_{14} = \frac{1}{4}S_{12}S_{34} + \frac{1}{4}V_{12\mu}V_{34}^\mu + \frac{1}{8}T_{12\mu\nu}T_{34}^{\mu\nu} + \frac{1}{4}P_{12}P_{34} - \frac{1}{4}A_{12\mu}A_{34}^\mu. \quad (205)$$

Week5 · TEX

4 Week 5

Conventions. Four-vector products are $p \cdot x = p^0t - \mathbf{p} \cdot \mathbf{x}$ (signature $(+, -, -, -)$), so that $e^{-ip \cdot x} = e^{-iE_p t + i\mathbf{p} \cdot \mathbf{x}}$ on shell.

$$E_p = \sqrt{|\mathbf{p}|^2 + m^2}.$$

4.1 Lecture 20: Quantization of the Real Scalar Field

Consider the real scalar field with

$$\mathcal{L} = \frac{1}{2}(\partial^\mu\phi)(\partial_\mu\phi) - \frac{m^2}{2}\phi^2, \quad \phi(\mathbf{x}, t) = \int \frac{d^3p}{(2\pi)^3} \tilde{\phi}(\mathbf{p}, t) e^{i\mathbf{p} \cdot \mathbf{x}}. \quad (206)$$

Let $\pi(\mathbf{x}, t)$ be the canonical momentum density conjugate to ϕ . The Klein–Gordon equation is $(\partial^\mu\partial_\mu + m^2)\phi(\mathbf{x}, t) = 0$. Since $\partial^\mu\partial_\mu = \partial_t^2 - \nabla^2$ and $\nabla^2 e^{i\mathbf{p} \cdot \mathbf{x}} = -|\mathbf{p}|^2 e^{i\mathbf{p} \cdot \mathbf{x}}$, substituting the Fourier decomposition gives, *for each momentum mode*,

$$\frac{d^2\tilde{\phi}(\mathbf{p}, t)}{dt^2} + (|\mathbf{p}|^2 + m^2)\tilde{\phi}(\mathbf{p}, t) = 0, \quad \omega_p^2 \equiv |\mathbf{p}|^2 + m^2. \quad (207)$$

Each mode is therefore a simple harmonic oscillator (SHO) of angular frequency $\omega_p = E_p$. In particular the mass is the angular frequency of the zero mode: $\omega_{\mathbf{p}=0} = m$.

Review of the single oscillator

For one oscillator with $H_{\text{SHO}} = \frac{1}{2}\tilde{\pi}^2 + \frac{1}{2}\omega^2\tilde{\phi}^2$ and $[\tilde{\phi}, \tilde{\pi}] = i$, set

$$\tilde{\phi} = \frac{1}{\sqrt{2\omega}} (a + a^\dagger), \quad \tilde{\pi} = -i\sqrt{\frac{\omega}{2}} (a - a^\dagger), \quad [a, a^\dagger] = 1. \quad (208)$$

Check of the commutator:

$$[\tilde{\phi}, \tilde{\pi}] = \frac{-i}{2} [a + a^\dagger, a - a^\dagger] = \frac{-i}{2} (-[a, a^\dagger] + [a^\dagger, a]) = \frac{-i}{2} (-2) = i. \quad (209)$$

Check of the Hamiltonian:

$$\begin{aligned} \frac{1}{2}\tilde{\pi}^2 &= -\frac{\omega}{4} (a^2 + a^{\dagger 2} - aa^\dagger - a^\dagger a), & \frac{1}{2}\omega^2\tilde{\phi}^2 &= \frac{\omega}{4} (a^2 + a^{\dagger 2} + aa^\dagger + a^\dagger a), \\ H_{\text{SHO}} &= \frac{\omega}{2} (aa^\dagger + a^\dagger a) = \omega \left(a^\dagger a + \frac{1}{2}\right). \end{aligned} \quad (210)$$

Mode expansion of the field

Promote each mode to an oscillator, $\omega \rightarrow E_p$, $a \rightarrow a_{\mathbf{p}}$, with continuum normalization

$$[a_{\mathbf{p}}, a_{\mathbf{q}}^\dagger] = (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}), \quad [a_{\mathbf{p}}, a_{\mathbf{q}}] = 0 = [a_{\mathbf{p}}^\dagger, a_{\mathbf{q}}^\dagger]. \quad (211)$$

Then, at fixed time,

$$\phi(\mathbf{x}, t) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} (a_{\mathbf{p}} e^{i\mathbf{p}\cdot\mathbf{x}} + a_{\mathbf{p}}^\dagger e^{-i\mathbf{p}\cdot\mathbf{x}}), \quad (212)$$

$$\pi(\mathbf{x}, t) = \int \frac{d^3p}{(2\pi)^3} (-i)\sqrt{\frac{E_p}{2}} (a_{\mathbf{p}} e^{i\mathbf{p}\cdot\mathbf{x}} - a_{\mathbf{p}}^\dagger e^{-i\mathbf{p}\cdot\mathbf{x}}). \quad (213)$$

(The time dependence is made explicit in the Heisenberg picture below; consistently $\pi = \dot{\phi}$.)

Equal-time canonical commutation relations

We look for $[\phi(\mathbf{x}, t), \pi(\mathbf{y}, t)]$:

$$[\phi(\mathbf{x}, t), \pi(\mathbf{y}, t)] = \int \frac{d^3p d^3q}{(2\pi)^6} \frac{(-i)}{\sqrt{2E_p}} \sqrt{\frac{E_q}{2}} [a_{\mathbf{p}} e^{i\mathbf{p}\cdot\mathbf{x}} + a_{\mathbf{p}}^\dagger e^{-i\mathbf{p}\cdot\mathbf{x}}, a_{\mathbf{q}} e^{i\mathbf{q}\cdot\mathbf{y}} - a_{\mathbf{q}}^\dagger e^{-i\mathbf{q}\cdot\mathbf{y}}].$$

Expanding the commutator and using $[a, a] = [a^\dagger, a^\dagger] = 0$,

$$\begin{aligned} [\dots] &= -[a_{\mathbf{p}}, a_{\mathbf{q}}^\dagger] e^{i\mathbf{p}\cdot\mathbf{x} - i\mathbf{q}\cdot\mathbf{y}} + [a_{\mathbf{p}}^\dagger, a_{\mathbf{q}}] e^{-i\mathbf{p}\cdot\mathbf{x} + i\mathbf{q}\cdot\mathbf{y}} \\ &= -(2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}) (e^{i\mathbf{p}\cdot\mathbf{x} - i\mathbf{q}\cdot\mathbf{y}} + e^{-i\mathbf{p}\cdot\mathbf{x} + i\mathbf{q}\cdot\mathbf{y}}), \end{aligned} \quad (214)$$

where $[a_{\mathbf{p}}^\dagger, a_{\mathbf{q}}] = -(2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q})$ was used.

Doing the $\mathbf{q}$ integral (so $\sqrt{E_q/E_p} \rightarrow 1$),

$$[\phi(\mathbf{x}, t), \pi(\mathbf{y}, t)] = \int \frac{d^3p}{(2\pi)^3} \frac{i}{2} (e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} + e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}) = i \delta^{(3)}(\mathbf{x} - \mathbf{y}), \quad (215)$$

since $\int \frac{d^3p}{(2\pi)^3} e^{\pm i\mathbf{p}\cdot\mathbf{r}} = \delta^{(3)}(\mathbf{r})$.

$$[\phi(\mathbf{x}, t), \pi(\mathbf{y}, t)] = i \delta^{(3)}(\mathbf{x} - \mathbf{y}), \quad [\phi(\mathbf{x}, t), \phi(\mathbf{y}, t)] = 0 = [\pi(\mathbf{x}, t), \pi(\mathbf{y}, t)]. \quad (216)$$

The last two follow the same way. For example

$$[\phi(\mathbf{x}), \phi(\mathbf{y})] = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} (e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} - e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}) = 0, \quad (217)$$

because the second term maps to minus the first under $\mathbf{p} \rightarrow -\mathbf{p}$ (E_p is even in $\mathbf{p}$).

The Hamiltonian

For a Lagrangian density $\mathcal{L}$, $\mathcal{H} = \pi \dot{\phi} - \mathcal{L}$. Here

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \frac{\partial}{\partial(\partial_\mu \phi)} \left(\frac{1}{2} \eta^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi \right) = \frac{1}{2} \left(\eta^{\mu\beta} \partial_\beta \phi + \eta^{\alpha\mu} \partial_\alpha \phi \right) = \partial^\mu \phi, \quad (218)$$

so $\pi = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} = \partial^0 \phi = \dot{\phi}$. With $\mathcal{L} = \frac{1}{2} \dot{\phi}^2 - \frac{1}{2} (\nabla \phi)^2 - \frac{1}{2} m^2 \phi^2$,

$$H = \int d^3x \left[\dot{\phi} \dot{\phi} - \mathcal{L} \right] = \int d^3x \left[\underbrace{\frac{1}{2} \pi^2}_{T_1} + \underbrace{\frac{1}{2} (\nabla \phi) \cdot (\nabla \phi)}_{T_2} + \underbrace{\frac{1}{2} m^2 \phi^2}_{T_3} \right]. \quad (219)$$

We need the spatial integrals $\int d^3x e^{i(\mathbf{p}+\mathbf{q}) \cdot \mathbf{x}} = (2\pi)^3 \delta^{(3)}(\mathbf{p} + \mathbf{q})$ and $\int d^3x e^{i(\mathbf{p}-\mathbf{q}) \cdot \mathbf{x}} = (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q})$.

T_1 . Using the mode expansion of π and $(-i)^2 = -1$,

$$\begin{aligned} \int d^3x \frac{1}{2} \pi^2 &= -\frac{1}{4} \int \frac{d^3p}{(2\pi)^3} E_p \left[a_{\mathbf{p}} a_{-\mathbf{p}} - a_{\mathbf{p}} a_{\mathbf{p}}^\dagger - a_{\mathbf{p}}^\dagger a_{\mathbf{p}} + a_{\mathbf{p}}^\dagger a_{-\mathbf{p}}^\dagger \right] \\ &= \int \frac{d^3p}{(2\pi)^3} \frac{E_p}{4} \left[a_{\mathbf{p}} a_{\mathbf{p}}^\dagger + a_{\mathbf{p}}^\dagger a_{\mathbf{p}} - a_{\mathbf{p}} a_{-\mathbf{p}} - a_{\mathbf{p}}^\dagger a_{-\mathbf{p}}^\dagger \right]. \end{aligned} \quad (220)$$

T_2 . Here $\nabla \phi = \int \frac{d^3p}{(2\pi)^3} \frac{i\mathbf{p}}{\sqrt{2E_p}} \left(a_{\mathbf{p}} e^{i\mathbf{p} \cdot \mathbf{x}} - a_{\mathbf{p}}^\dagger e^{-i\mathbf{p} \cdot \mathbf{x}} \right)$. The product $(i\mathbf{p}) \cdot (i\mathbf{q}) = -\mathbf{p} \cdot \mathbf{q}$ equals $+\mathbf{p}^2$ when $\mathbf{q} = -\mathbf{p}$ and $-\mathbf{p}^2$ when $\mathbf{q} = \mathbf{p}$, so

$$\begin{aligned} \int d^3x \frac{1}{2} (\nabla \phi)^2 &= \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \left[|\mathbf{p}|^2 a_{\mathbf{p}} a_{-\mathbf{p}} + |\mathbf{p}|^2 a_{\mathbf{p}} a_{\mathbf{p}}^\dagger + |\mathbf{p}|^2 a_{\mathbf{p}}^\dagger a_{\mathbf{p}} + |\mathbf{p}|^2 a_{\mathbf{p}}^\dagger a_{-\mathbf{p}}^\dagger \right] \\ &= \int \frac{d^3p}{(2\pi)^3} \frac{|\mathbf{p}|^2}{4E_p} \left[a_{\mathbf{p}} a_{\mathbf{p}}^\dagger + a_{\mathbf{p}}^\dagger a_{\mathbf{p}} + a_{\mathbf{p}} a_{-\mathbf{p}} + a_{\mathbf{p}}^\dagger a_{-\mathbf{p}}^\dagger \right]. \end{aligned} \quad (221)$$

T_3 . Similarly, with ϕ^2 producing the same four terms with all plus signs,

$$\int d^3x \frac{1}{2} m^2 \phi^2 = \int \frac{d^3p}{(2\pi)^3} \frac{m^2}{4E_p} \left[a_{\mathbf{p}} a_{\mathbf{p}}^\dagger + a_{\mathbf{p}}^\dagger a_{\mathbf{p}} + a_{\mathbf{p}} a_{-\mathbf{p}} + a_{\mathbf{p}}^\dagger a_{-\mathbf{p}}^\dagger \right]. \quad (222)$$

Sum. Since $|\mathbf{p}|^2 + m^2 = E_p^2$, the coefficient in $T_2 + T_3$ is $\frac{E_p^2}{4E_p} = \frac{E_p}{4}$. Adding T_1 , the $a_{\mathbf{p}} a_{-\mathbf{p}}$ and $a_{\mathbf{p}}^\dagger a_{-\mathbf{p}}^\dagger$ terms cancel (equivalently $\frac{|\mathbf{p}|^2 + m^2}{2E_p} - \frac{E_p}{2} = 0$ in the notation of the handwritten notes), leaving

$$H = \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} E_p \left(a_{\mathbf{p}} a_{\mathbf{p}}^\dagger + a_{\mathbf{p}}^\dagger a_{\mathbf{p}} \right). \quad (223)$$

Using $a_{\mathbf{p}} a_{\mathbf{p}}^\dagger = a_{\mathbf{p}}^\dagger a_{\mathbf{p}} + (2\pi)^3 \delta^{(3)}(0)$,

$$H = \int \frac{d^3p}{(2\pi)^3} E_p \left(a_{\mathbf{p}}^\dagger a_{\mathbf{p}} + \frac{(2\pi)^3}{2} \delta^{(3)}(0) \right). \quad (224)$$

4.2 Lecture 21: The Vacuum, Normal Ordering and Lorentz-Invariant States

The single oscillator has $H_{\text{SHO}} = \hbar\omega(a^\dagger a + \frac{1}{2})$ and ground state $a|0\rangle = 0$ with energy $\hbar\omega/2$. For the field, the ground state satisfies $a_{\mathbf{p}}|0\rangle = 0$ for all $\mathbf{p}$, hence $a_{\mathbf{p}}^\dagger a_{\mathbf{p}}|0\rangle = 0 \forall \mathbf{p}$, and

$$|0\rangle = \prod_{\mathbf{p}} |0\rangle_{\mathbf{p}}. \quad (225)$$

For two degrees of freedom the analogue is $|0\rangle = |0\rangle_1 \otimes |0\rangle_2$ with $a_i|0\rangle = 0$. The state $a_{\mathbf{p}}^\dagger|0\rangle$ has one quantum in mode $\mathbf{p}$ and the vacuum in all other modes:

$$a_{\mathbf{p}}^\dagger|0\rangle = |1\rangle_{\mathbf{p}} \otimes \prod_{\mathbf{q} \neq \mathbf{p}} |0\rangle_{\mathbf{q}} \propto |\mathbf{p}\rangle. \quad (226)$$

Acting with H on the vacuum gives the zero-point energy

$$H|0\rangle = \left[\int \frac{d^3p}{(2\pi)^3} \frac{E_p}{2} (2\pi)^3 \delta^{(3)}(0) \right] |0\rangle, \quad (227)$$

which is infinite. Since $(2\pi)^3 \delta^{(3)}(0) = \int d^3x = V$, this is $V \int \frac{d^3p}{(2\pi)^3} \frac{E_p}{2}$, divergent in both volume and momentum. Only energy differences are measurable, so we *define* $H|0\rangle = 0$ by *normal ordering* (all $a^\dagger$ to the left of all a):

$$:H: = \int \frac{d^3p}{(2\pi)^3} E_p a_{\mathbf{p}}^\dagger a_{\mathbf{p}}. \quad (228)$$

From here on H means $:H:$. The vacuum is normalized, $\langle 0|0\rangle = \prod_{\mathbf{p}} \langle 0|0\rangle_{\mathbf{p}} = 1$, and is boost invariant, $U(\Lambda)|0\rangle = |0\rangle$ (the vacuum is a product over all modes, and a boost only relabels $\mathbf{p} \rightarrow \mathbf{p}'$).

One-particle states are not Lorentz invariantly normalized

Define $|\mathbf{p}\rangle \propto a_{\mathbf{p}}^\dagger|0\rangle$. Then

$$\langle \mathbf{p}|\mathbf{q}\rangle = \langle 0|a_{\mathbf{p}}a_{\mathbf{q}}^\dagger|0\rangle = \langle 0|(a_{\mathbf{q}}^\dagger a_{\mathbf{p}} + (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}))|0\rangle = (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}). \quad (229)$$

This is *not* Lorentz invariant. To see this take a boost along z with velocity β , $p_z \rightarrow p'_z = \gamma(p_z + \beta E_p)$, $E_p \rightarrow E'_p = \gamma(E_p + \beta p_z)$. The transverse components are unchanged, so only $\delta(p_z - q_z)$ changes: since $\delta(f(p_z) - f(q_z)) = \delta(p_z - q_z)/|f'|$,

$$\delta(p_z - q_z) = \frac{dp'_z}{dp_z} \delta(p'_z - q'_z), \quad \frac{dp'_z}{dp_z} = \gamma \left(1 + \beta \frac{dE_p}{dp_z} \right) = \gamma \left(1 + \beta \frac{p_z}{E_p} \right) = \frac{E'_p}{E_p}. \quad (230)$$

Hence $\delta^{(3)}(\mathbf{p} - \mathbf{q}) = \frac{E'_p}{E_p} \delta^{(3)}(\mathbf{p}' - \mathbf{q}')$, i.e.

$$E_p \delta^{(3)}(\mathbf{p} - \mathbf{q}) = E'_p \delta^{(3)}(\mathbf{p}' - \mathbf{q}') \quad \text{is Lorentz invariant.} \quad (231)$$

So we define the relativistically normalized states

$$|\mathbf{p}\rangle = \sqrt{2E_p} a_{\mathbf{p}}^\dagger|0\rangle, \quad \langle \mathbf{p}|\mathbf{q}\rangle = 2E_p (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}). \quad (232)$$

[Correction/note:] in the handwritten notes the right-hand side of $\sqrt{2E_p} a_{\mathbf{p}}^\dagger|0\rangle$ is written as $|0\rangle$; it should be $|\mathbf{p}\rangle$.

Commutator with H and the action of a boost

$$\begin{aligned} [H, a_{\mathbf{p}}] &= \int \frac{d^3q}{(2\pi)^3} E_q [a_{\mathbf{q}}^\dagger a_{\mathbf{q}}, a_{\mathbf{p}}] = \int \frac{d^3q}{(2\pi)^3} E_q \left(a_{\mathbf{q}}^\dagger [a_{\mathbf{q}}, a_{\mathbf{p}}] + [a_{\mathbf{q}}^\dagger, a_{\mathbf{p}}] a_{\mathbf{q}} \right) \\ &= - \int \frac{d^3q}{(2\pi)^3} E_q (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}) a_{\mathbf{q}} = -E_p a_{\mathbf{p}}, \end{aligned} \quad (233)$$

and likewise $[H, a_{\mathbf{p}}^\dagger] = +E_p a_{\mathbf{p}}^\dagger$. Hence $H |\mathbf{p}\rangle = E_p |\mathbf{p}\rangle$.

A boost acts as $\mathbf{p} \rightarrow \mathbf{p}' = \Lambda \mathbf{p}$ with $U(\Lambda) |\mathbf{p}\rangle = |\Lambda \mathbf{p}\rangle$. Then

$$U(\Lambda) \sqrt{2E_p} a_{\mathbf{p}}^\dagger |0\rangle = |\Lambda \mathbf{p}\rangle \implies U(\Lambda) \sqrt{2E_p} a_{\mathbf{p}}^\dagger U^{-1}(\Lambda) U(\Lambda) |0\rangle = \sqrt{2E_{\Lambda p}} a_{\Lambda \mathbf{p}}^\dagger |0\rangle, \quad (234)$$

and with $U(\Lambda) |0\rangle = |0\rangle$ this gives $U(\Lambda) a_{\mathbf{p}}^\dagger U^{-1}(\Lambda) = \sqrt{E_{\Lambda p}/E_p} a_{\Lambda \mathbf{p}}^\dagger$.

Invariant measure

d^4p and $p^2 - m^2$ are Lorentz invariant, so $\int \frac{d^4p}{(2\pi)^4} \delta(p^2 - m^2)$ is invariant. Evaluating the p^0 integral with $\delta(g(p^0)) = \sum_i \delta(p^0 - p_i^0)/|g'(p_i^0)|$ and $g' = 2p^0$,

$$\begin{aligned} \int \frac{d^4p}{(2\pi)^4} \delta(p^2 - m^2) &= \int \frac{d^3p}{(2\pi)^4} \int dp^0 \delta(p_0^2 - E_p^2) = \int \frac{d^3p}{(2\pi)^4} \int dp^0 \frac{\delta(p^0 - E_p) + \delta(p^0 + E_p)}{2|p^0|} \\ &= \int \frac{d^3p}{(2\pi)^4} \frac{1}{2E_p} [\text{positive energy}] + [\text{negative energy}]. \end{aligned} \quad (235)$$

Restricting to $p^0 > 0$ shows that $\frac{d^3p}{(2\pi)^3 2E_p}$ is a Lorentz-invariant measure. This is why $2E_p$ appears in the normalization above.

4.3 Lecture 22: Heisenberg Picture and the Propagator

Time dependence of the fields

In the Heisenberg picture $\phi(x) = e^{iHt} \phi(\mathbf{x}) e^{-iHt}$. From $[H, a_{\mathbf{p}}] = -E_p a_{\mathbf{p}}$,

$$H a_{\mathbf{p}} = a_{\mathbf{p}} H - E_p a_{\mathbf{p}} = a_{\mathbf{p}} (H - E_p), \quad H^2 a_{\mathbf{p}} = H a_{\mathbf{p}} (H - E_p) = a_{\mathbf{p}} (H - E_p)^2, \quad (236)$$

and in general $H^n a_{\mathbf{p}} = a_{\mathbf{p}} (H - E_p)^n$, so $e^{iHt} a_{\mathbf{p}} = a_{\mathbf{p}} e^{i(H-E_p)t}$. Therefore

$$e^{iHt} a_{\mathbf{p}} e^{-iHt} = a_{\mathbf{p}} e^{i(H-E_p)t} e^{-iHt} = a_{\mathbf{p}} e^{-iE_p t}, \quad e^{iHt} a_{\mathbf{p}}^\dagger e^{-iHt} = a_{\mathbf{p}}^\dagger e^{+iE_p t}. \quad (237)$$

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(a_{\mathbf{p}} e^{-ip \cdot x} + a_{\mathbf{p}}^\dagger e^{ip \cdot x} \right) \Big|_{p^0=E_p}. \quad (238)$$

The two-point amplitude

Define states created by the field at a spacetime point, $|x\rangle = \phi(x) |0\rangle$ and $|y\rangle = \phi(y) |0\rangle$, and take $y^0 > x^0$ for time ordering. Then

$$\begin{aligned} \langle y|x \rangle &= \langle 0| \phi(y) \phi(x) |0\rangle \\ &= \int \frac{d^3p d^3q}{(2\pi)^6} \frac{1}{\sqrt{2E_p 2E_q}} \langle 0| \left(a_{\mathbf{p}} e^{-ip \cdot y} + a_{\mathbf{p}}^\dagger e^{ip \cdot y} \right) \left(a_{\mathbf{q}} e^{-iq \cdot x} + a_{\mathbf{q}}^\dagger e^{iq \cdot x} \right) |0\rangle \\ &= \int \frac{d^3p d^3q}{(2\pi)^6} \frac{1}{\sqrt{2E_p 2E_q}} \langle 0| a_{\mathbf{p}} a_{\mathbf{q}}^\dagger |0\rangle e^{-ip \cdot y + iq \cdot x}, \end{aligned} \quad (239)$$

because the other three terms vanish ($a_{\mathbf{q}} |0\rangle = 0$, $\langle 0| a_{\mathbf{p}}^\dagger = 0$). With $\langle 0| a_{\mathbf{p}} a_{\mathbf{q}}^\dagger |0\rangle = (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q})$,

$$\langle y|x \rangle = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} e^{-ip \cdot (y-x)} \quad (y^0 > x^0). \quad (240)$$

This is the amplitude for a particle created at x to propagate to y .

Case 1: timelike separation, $(x - y)^2 > 0$. Choose a frame with $y - x = (t, \mathbf{0})$:

$$\langle y|x \rangle = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} e^{-iE_p t}. \quad (241)$$

For large t the phase is stationary at $\mathbf{p} = 0$ (the method of stationary phase). Expanding $E_p \simeq m + |\mathbf{p}|^2/2m$ and using $\int d^3p e^{-i|\mathbf{p}|^2 t/2m} = (2\pi m/it)^{3/2}$,

$$\langle y|x \rangle \simeq \frac{\sqrt{m}}{2} (2\pi it)^{-3/2} e^{-imt} \sim t^{-3/2} e^{-imt}. \quad (242)$$

So for timelike separation the amplitude is a pure phase e^{-imt} times a slowly decaying power.

Case 2: spacelike separation, $(x - y)^2 < 0$. A suitable frame has $y - x = (0, \mathbf{r})$, $r = |\mathbf{r}|$. Then $-p \cdot (y - x) = +\mathbf{p} \cdot \mathbf{r}$ and, using spherical coordinates with the polar axis along $\mathbf{r}$,

$$\begin{aligned} \langle y|x \rangle &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} e^{i\mathbf{p} \cdot \mathbf{r}} = \int \frac{p^2 dp d\varphi d\cos\theta}{(2\pi)^3} \frac{1}{2E_p} e^{ipr \cos\theta} \\ &= \frac{2\pi}{(2\pi)^3} \int_0^\infty p^2 dp \frac{1}{2E_p} \frac{e^{ipr} - e^{-ipr}}{ipr} = \frac{1}{(2\pi)^2} \frac{1}{2ir} \int_0^\infty dp \frac{p(e^{ipr} - e^{-ipr})}{E_p} \\ &= \frac{1}{(2\pi)^2 (2ir)} \int_{-\infty}^\infty dp \frac{p e^{ipr}}{\sqrt{p^2 + m^2}}, \end{aligned} \quad (243)$$

where the second term was mapped to $\int_{-\infty}^0$ by $p \rightarrow -p$.

The integrand has branch points at $p = \pm im$. Deform the contour into the upper half plane so that it wraps the cut along $p = i\rho$, $\rho \in [m, \infty)$, where $\sqrt{p^2 + m^2} = \pm i\sqrt{\rho^2 - m^2}$ on the two sides of the cut. This gives

$$\langle y|x \rangle = \frac{1}{4\pi^2 r} \int_m^\infty d\rho \frac{\rho e^{-\rho r}}{\sqrt{\rho^2 - m^2}}. \quad (244)$$

As a check, for $m = 0$ this is $\frac{1}{4\pi^2 r} \int_0^\infty e^{-\rho r} d\rho = \frac{1}{4\pi^2 r^2}$, which agrees with the direct result $\frac{1}{4\pi^2 r} \int_0^\infty \sin(\rho r) d\rho = \frac{1}{4\pi^2 r^2}$. For large r the integral is dominated by $\rho \simeq m$ (steepest descent). Setting $\rho = m + u$ and $\sqrt{\rho^2 - m^2} \simeq \sqrt{2mu}$,

$$\int_m^\infty \frac{\rho e^{-\rho r}}{\sqrt{\rho^2 - m^2}} d\rho \simeq e^{-mr} \sqrt{\frac{m}{2}} \int_0^\infty \frac{e^{-ur}}{\sqrt{u}} du = e^{-mr} \sqrt{\frac{\pi m}{2r}}, \quad (245)$$

so $\langle y|x \rangle \sim e^{-mr}$ (times $r^{-3/2}$): exponentially small outside the light cone but not zero. (The exact result is $\frac{m}{4\pi^2 r} K_1(mr)$.) Compare: for timelike separation the amplitude is a phase and does not decay exponentially.

The commutator of fields and causality

$$\langle 0 | [\phi(x), \phi(y)] | 0 \rangle = \langle x|y \rangle - \langle y|x \rangle. \quad (246)$$

For *timelike* separation with $y - x = (t, \mathbf{0})$, $t > 0$: $\langle y|x \rangle \sim e^{-imt}$ while $\langle x|y \rangle = \langle y|x \rangle^* \sim e^{+imt}$ (it is the complex conjugate), so the commutator is $\sim e^{imt} - e^{-imt} \neq 0$.

For *spacelike* separation both terms are the same function of $r = |\mathbf{r}|$, since $\langle y|x \rangle$ depends only on r (for $y - x = (0, \mathbf{r})$ and $x - y = (0, -\mathbf{r})$): $e^{-mr} - e^{-mr} = 0$. Hence

$$\langle 0 | [\phi(x), \phi(y)] | 0 \rangle = 0 \quad \text{for } (x - y)^2 < 0 \quad (\text{causality}). \quad (247)$$

In general (for any separation), using $\langle x | y \rangle = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_p} e^{-ip \cdot (x-y)}$,

$$\langle 0 | [\phi(x), \phi(y)] | 0 \rangle = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_p} \left(e^{-ip \cdot (x-y)} - e^{ip \cdot (x-y)} \right). \quad (248)$$

In the second term send $\mathbf{p} \rightarrow -\mathbf{p}$ and then define the four-vector $p^\mu = (-E_p, \mathbf{p})$, so that $e^{ip \cdot (x-y)}|_{\mathbf{p} \rightarrow -\mathbf{p}} = e^{-ip \cdot (x-y)}|_{p^0 = -E_p}$. Then

$$\begin{aligned} \langle 0 | [\phi(x), \phi(y)] | 0 \rangle &= \int \frac{d^3 p}{(2\pi)^3} \left[\frac{e^{-ip \cdot (x-y)}}{2E_p} \Big|_{p^0 = E_p} + \frac{e^{-ip \cdot (x-y)}}{-2E_p} \Big|_{p^0 = -E_p} \right] \\ &= \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2\pi i} \oint dp^0 \frac{e^{-ip \cdot (x-y)}}{(p^0 - E_p)(p^0 + E_p)}, \end{aligned} \quad (249)$$

where the contour encircles both poles counterclockwise (the residues at $p^0 = \pm E_p$ are $\pm e^{-ip \cdot (x-y)} / 2E_p$), and $p_0^2 - E_p^2 = p^2 - m^2$.

4.4 Lecture 23: Green's Functions and the Feynman Propagator

The commutator above is the common building block. For $x^0 > y^0$ define the commutator as

$$\langle 0 | [\phi(x), \phi(y)] | 0 \rangle = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_p} \left(e^{-ip \cdot (x-y)} - e^{ip \cdot (x-y)} \right). \quad (250)$$

Retarded Green's function. Take the contour in the p^0 plane *below* the poles:

$$D_R(x - y) = \int \frac{d^4 p}{(2\pi)^4} \frac{i e^{-ip \cdot (x-y)}}{(p^0 + i\epsilon)^2 - E_p^2}, \quad (251)$$

with poles at $p^0 = \pm E_p - i\epsilon$ (lower half plane). For $x^0 < y^0$ close the contour in the upper half plane ($e^{-ip^0 t}$ decays for $\text{Im } p^0 > 0$ when $t < 0$): no poles are enclosed, so $D_R = 0$. For $x^0 > y^0$ close in the lower half plane (clockwise, giving $-2\pi i \sum \text{Res}$):

$$\int \frac{dp^0}{2\pi} i(-2\pi i) \sum \text{Res} = \frac{e^{-iE_p t}}{2E_p} + \frac{e^{iE_p t}}{-2E_p}, \quad t = x^0 - y^0, \quad (252)$$

so

$$D_R(x - y) = \theta(x^0 - y^0) \langle 0 | [\phi(x), \phi(y)] | 0 \rangle. \quad (253)$$

Advanced Green's function. Take the contour *above* the poles,

$$D_A(x - y) = \int \frac{d^4 p}{(2\pi)^4} \frac{i e^{-ip \cdot (x-y)}}{(p^0 - i\epsilon)^2 - E_p^2}, \quad (254)$$

poles at $p^0 = \pm E_p + i\epsilon$. For $x^0 > y^0$ closing below encloses nothing, so $D_A = 0$. For $x^0 < y^0$ closing above (counterclockwise) gives $D_A = -\theta(y^0 - x^0) \langle 0 | [\phi(x), \phi(y)] | 0 \rangle = \theta(y^0 - x^0) \langle 0 | [\phi(y), \phi(x)] | 0 \rangle$.

Feynman propagator. Now take poles displaced symmetrically, at $p^0 = \pm(E_p - i\epsilon)$: the positive-energy pole lies below the real axis and the negative-energy pole above. Then both $\langle y|x \rangle$ (for $y^0 > x^0$) and $\langle x|y \rangle$ (for $x^0 > y^0$) contribute:

$$D_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i e^{-ip \cdot (x-y)}}{p_0^2 - (E_p - i\epsilon)^2} = \int \frac{d^4p}{(2\pi)^4} \frac{i e^{-ip \cdot (x-y)}}{p^2 - m^2 + i\epsilon'}, \quad (255)$$

where $(E_p - i\epsilon)^2 = E_p^2 - 2i\epsilon E_p + O(\epsilon^2)$ defines $\epsilon' = 2\epsilon E_p > 0$. Evaluate the p^0 integral. For $t = x^0 - y^0 > 0$ close below, picking only $p^0 = E_p - i\epsilon$ (clockwise):

$$\int \frac{dp^0}{2\pi} \frac{i e^{-ip^0 t}}{(p^0 - E_p + i\epsilon)(p^0 + E_p - i\epsilon)} = i(-i) \frac{e^{-iE_p t}}{2E_p} = \frac{e^{-iE_p t}}{2E_p}, \quad (256)$$

so $D_F = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} e^{-ip \cdot (x-y)} = \langle 0| \phi(x) \phi(y) |0 \rangle$. For $t < 0$ close above, picking $p^0 = -E_p + i\epsilon$ (counterclockwise): $i \cdot i \frac{e^{iE_p t}}{-2E_p} = \frac{e^{iE_p t}}{2E_p}$, and after $\mathbf{p} \rightarrow -\mathbf{p}$ this is $\langle 0| \phi(y) \phi(x) |0 \rangle$. Let T be the time-ordering operator. Then

$$\langle 0| T \phi(x) \phi(y) |0 \rangle = \theta(x^0 - y^0) \overline{\langle 0| \phi(x) \phi(y) |0 \rangle} + \theta(y^0 - x^0) \langle 0| \phi(y) \phi(x) |0 \rangle = D_F(x-y). \quad (257)$$

This is written $\overline{\phi(x) \phi(y)} \equiv D_F(x-y)$ and called a *contraction*, to be used in Wick's theorem later in the course.

Since $(\partial^\mu \partial_\mu + m^2)e^{-ip \cdot x} = (m^2 - p^2)e^{-ip \cdot x}$, the Green's functions obey

$$(\partial^\mu \partial_\mu + m^2)D_F(x-y) = \int \frac{d^4p}{(2\pi)^4} i \frac{-(p^2 - m^2)}{p^2 - m^2} e^{-ip \cdot (x-y)} = -i \delta^{(4)}(x-y), \quad (258)$$

and the same equation holds for D_R and D_A .

Several scalar fields

For n real fields with different masses,

$$\mathcal{L} = \sum_{i=1}^n \left[\frac{1}{2} (\partial^\mu \phi_i) (\partial_\mu \phi_i) - \frac{1}{2} m_i^2 \phi_i^2 \right] = \frac{1}{2} (\partial^\mu \Phi)^\top (\partial_\mu \Phi) - \frac{1}{2} \Phi^\top M^2 \Phi, \quad (259)$$

with $\Phi = (\phi_1, \dots, \phi_n)^\top$ and $M^2 = \text{diag}(m_1^2, \dots, m_n^2)$. The fields decouple, each with its own oscillators $a_{i\mathbf{p}}$, so

$$\phi_i(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p^{(i)}}} \left[a_{i\mathbf{p}} e^{-ip \cdot x} + a_{i\mathbf{p}}^\dagger e^{ip \cdot x} \right], \quad E_p^{(i)} = \sqrt{|\mathbf{p}|^2 + m_i^2}, \quad (260)$$

$$\langle y, j | x, i \rangle \propto \delta_{ij}, \quad \langle 0| T \phi_j(y) \phi_i(x) |0 \rangle = \delta_{ij} D_F^{(m_i)}(y-x), \quad (261)$$

where $D_F^{(m_i)}$ is the Feynman propagator with mass m_i .

4.5 Lecture 24: The Complex Scalar Field

Instead of two real fields ϕ_1, ϕ_2 we use the complex linear combination

$$\phi = \frac{\phi_1 + i\phi_2}{\sqrt{2}}, \quad \phi^\dagger = \frac{\phi_1 - i\phi_2}{\sqrt{2}}. \quad (262)$$

With equal masses,

$$\mathcal{L} = \frac{1}{2}(\partial^\mu \phi_1)(\partial_\mu \phi_1) - \frac{m^2}{2}\phi_1^2 + \frac{1}{2}(\partial^\mu \phi_2)(\partial_\mu \phi_2) - \frac{m^2}{2}\phi_2^2 = (\partial^\mu \phi)^\dagger(\partial_\mu \phi) - m^2 \phi^\dagger \phi, \quad (263)$$

since $\phi^\dagger \phi = \frac{1}{2}(\phi_1^2 + \phi_2^2)$ and $(\partial\phi)^\dagger(\partial\phi) = \frac{1}{2}[(\partial\phi_1)^2 + (\partial\phi_2)^2]$. Each real field has the expansion $\phi_i(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} [a_{i\mathbf{p}} e^{-ip \cdot x} + a_{i\mathbf{p}}^\dagger e^{ip \cdot x}]$, hence

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left[\frac{a_{1\mathbf{p}} + ia_{2\mathbf{p}}}{\sqrt{2}} e^{-ip \cdot x} + \frac{a_{1\mathbf{p}}^\dagger + ia_{2\mathbf{p}}^\dagger}{\sqrt{2}} e^{ip \cdot x} \right]. \quad (264)$$

Define

$$a_{\mathbf{p}} = \frac{a_{1\mathbf{p}} + ia_{2\mathbf{p}}}{\sqrt{2}}, \quad b_{\mathbf{p}} = \frac{a_{1\mathbf{p}} - ia_{2\mathbf{p}}}{\sqrt{2}} \implies b_{\mathbf{p}}^\dagger = \frac{a_{1\mathbf{p}}^\dagger + ia_{2\mathbf{p}}^\dagger}{\sqrt{2}}. \quad (265)$$

Then

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} [a_{\mathbf{p}} e^{-ip \cdot x} + b_{\mathbf{p}}^\dagger e^{ip \cdot x}], \quad \phi^\dagger(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} [b_{\mathbf{p}} e^{-ip \cdot x} + a_{\mathbf{p}}^\dagger e^{ip \cdot x}]. \quad (266)$$

The commutators follow from $[a_{i\mathbf{p}}, a_{j\mathbf{q}}^\dagger] = \delta_{ij}(2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q})$:

$$\begin{aligned} [a_{\mathbf{p}}, a_{\mathbf{q}}^\dagger] &= \frac{1}{2} ([a_1, a_1^\dagger] + [a_2, a_2^\dagger]) = (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}), \\ [b_{\mathbf{p}}, b_{\mathbf{q}}^\dagger] &= \frac{1}{2} ([a_1, a_1^\dagger] + [a_2, a_2^\dagger]) = (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}), \\ [a_{\mathbf{p}}, b_{\mathbf{q}}] &= \frac{1}{2} ([a_1, a_1] - [a_2, a_2]) = 0, \quad [a_{\mathbf{p}}, b_{\mathbf{q}}^\dagger] = \frac{1}{2} ([a_1, a_1^\dagger] - [a_2, a_2^\dagger]) = 0. \end{aligned} \quad (267)$$

The states are $|\mathbf{p}\rangle_i = \sqrt{2E_p} a_{i\mathbf{p}}^\dagger |0\rangle$, and

$$\frac{|\mathbf{p}\rangle_1 - i|\mathbf{p}\rangle_2}{\sqrt{2}} = \sqrt{2E_p} a_{\mathbf{p}}^\dagger |0\rangle \equiv |\mathbf{p}\rangle \text{ (particle)}, \quad \frac{|\mathbf{p}\rangle_1 + i|\mathbf{p}\rangle_2}{\sqrt{2}} = \sqrt{2E_p} b_{\mathbf{p}}^\dagger |0\rangle \equiv |\bar{\mathbf{p}}\rangle \text{ (antiparticle)}. \quad (268)$$

Propagation of particle and antiparticle

ϕ contains $b^\dagger$ and $\phi^\dagger$ contains $a^\dagger$. Hence $\phi(x)|0\rangle$ creates an antiparticle at x , while $\phi^\dagger(x)|0\rangle$ creates a particle. Write $|\bar{x}\rangle = \phi(x)|0\rangle$, $|\bar{y}\rangle = \phi(y)|0\rangle$ and $|x\rangle = \phi^\dagger(x)|0\rangle$, $|y\rangle = \phi^\dagger(y)|0\rangle$. Then

$$\langle \bar{y} | \bar{x} \rangle = \langle 0 | \phi^\dagger(y) \phi(x) | 0 \rangle \quad (\text{antiparticle propagating from } x \text{ to } y), \quad (269)$$

$$\langle y | x \rangle = \langle 0 | \phi(y) \phi^\dagger(x) | 0 \rangle \quad (\text{particle propagating from } x \text{ to } y). \quad (270)$$

Fourier-space solution of the Klein–Gordon equation

Both ϕ and ϕ^* satisfy $(\square + m^2)\phi = 0$ and $(\square + m^2)\phi^* = 0$. Write $\phi(t, \mathbf{x}) = \int \frac{d^3p}{(2\pi)^3} e^{i\mathbf{p} \cdot \mathbf{x}} \tilde{\phi}(t, \mathbf{p})$, so the Fourier mode obeys $\ddot{\tilde{\phi}} + E_p^2 \tilde{\phi} = 0$ with general solution

$$\tilde{\phi}(t, \mathbf{p}) = A_1(\mathbf{p}) e^{-iE_p t} + B_1(\mathbf{p}) e^{iE_p t}. \quad (271)$$

Let $f(t, \mathbf{p}) \equiv \tilde{\phi}(t, \mathbf{p})$ and let $g(t, \mathbf{p})$ be the Fourier transform of ϕ^* , i.e. $\phi^*(t, \mathbf{x}) = \int \frac{d^3 p}{(2\pi)^3} e^{i\mathbf{p}\cdot\mathbf{x}} g(t, \mathbf{p})$. Taking the complex conjugate of the expansion of ϕ , $\phi^* = \int \frac{d^3 p}{(2\pi)^3} e^{-i\mathbf{p}\cdot\mathbf{x}} f^*(t, \mathbf{p}) = \int \frac{d^3 p}{(2\pi)^3} e^{i\mathbf{p}\cdot\mathbf{x}} f^*(t, -\mathbf{p})$, so

$$f^*(t, \mathbf{p}) = g(t, -\mathbf{p}). \quad (272)$$

The mode g also solves the oscillator equation,

$$g(t, \mathbf{p}) = A_2(\mathbf{p}) e^{-iE_p t} + B_2(\mathbf{p}) e^{iE_p t}. \quad (273)$$

[Correction/note:] the notes stop here. Completing the step: $g(t, \mathbf{p}) = f^*(t, -\mathbf{p}) = A_1^*(-\mathbf{p}) e^{iE_p t} + B_1^*(-\mathbf{p}) e^{-iE_p t}$, so comparing coefficients $A_2(\mathbf{p}) = B_1^*(-\mathbf{p})$ and $B_2(\mathbf{p}) = A_1^*(-\mathbf{p})$. Hence ϕ and ϕ^* are fixed by the two independent complex functions A_1, B_1 , consistent with the two independent operators $a_{\mathbf{p}}, b_{\mathbf{p}}$ above. Indeed, relabelling $\mathbf{p} \rightarrow -\mathbf{p}$ in the B_1 term, $\phi(x) = \int \frac{d^3 p}{(2\pi)^3} [A_1(\mathbf{p}) e^{-ip \cdot x} + B_1(-\mathbf{p}) e^{ip \cdot x}]$, which matches the expansion above with $A_1 = a_{\mathbf{p}}/\sqrt{2E_p}$ and $B_1(-\mathbf{p}) = b_{\mathbf{p}}^\dagger/\sqrt{2E_p}$.

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